

Claw-free graphs, skeletal graphs, and a stronger conjecture on ω , Δ , and χ

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Abstract

The second author's ω , Δ , χ conjecture proposes that every graph satisfies $\chi \leq \lceil \frac{1}{2}(\Delta + 1 + \omega) \rceil$. In this paper we prove that the conjecture holds for all claw-free graphs. Our approach uses the structure theorem of Chudnovsky and Seymour.

Along the way we discuss a stronger local conjecture, and prove that it holds for claw-free graphs with a three-colourable complement. To prove our results we introduce a very useful χ -preserving reduction on homogeneous pairs of cliques, and thus restrict our view to so-called *skeletal* graphs.

1 Introduction

In this paper the graphs we consider are simple, loopless, and finite. The multigraphs we consider are finite and may have loops. We say that a graph G is *claw-free* if it does not contain the complete bipartite graph $K_{1,3}$ as an induced subgraph, i.e. if no vertex of G has three mutually nonadjacent neighbours. Claw-free graphs are a natural generalization of line graphs and quasi-line graphs (which we define in Section 3), and have been the subject of substantial interest since Parthasarathy and Ravindra's proof of the Strong Perfect Graph Conjecture for claw-free graphs [28]. Chvátal and Sbihi [9] offered the first deep insight into the structure of claw-free graphs, proving a decomposition theorem for Berge claw-free graphs that was later refined by Maffray and Reed [24].

Chudnovsky and Seymour recently gave a refined description of the structure of all claw-free graphs [4]. Their structure theorems for claw-free graphs have led to a wealth of recent results, for example a new algorithm for the maximum-weight stable set problem [27] and new results on the stable set polytope [14, 18].

In this paper we give a new bound on the chromatic number $\chi(G)$ when G is claw-free. The bound is in terms of the maximum degree $\Delta(G)$ and the clique number $\omega(G)$.

Remark: Since we first proved these results, which appear in the first author's thesis [20], several related results have appeared, e.g. [2, 13]. To minimize the length of this paper we take advantage of this wherever possible.

1.1 ω , Δ , and χ

It is easy to show that $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$ for any graph. The second author conjectured that modulo a round-up, χ is closer to its trivial lower bound than its trivial upper bound [30]. We use $\gamma(G)$ to denote $\chi(G) \leq \lceil \frac{1}{2}(\Delta(G) + 1 + \omega(G)) \rceil$.

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Conjecture 1.1 (Reed). *For any graph G , $\chi(G) \leq \gamma(G)$.*

In 2008 the first author proposed a local strengthening of this conjecture [20]. Before stating it we introduce some more notation. For a vertex v , let $\tilde{N}(v)$ denote the closed neighbourhood of v , i.e. $\{v\} \cup N(v)$. For $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced on S . Let $\omega(v)$ denote the maximum size of a clique containing v , i.e. $\omega(G[\tilde{N}(v)])$. Finally, let $\gamma_\ell(v)$ denote $\gamma(G[\tilde{N}(v)])$ and let $\gamma_\ell(G)$ denote $\max_{v \in V(G)} \gamma_\ell(v)$.

Conjecture 1.2 (King). *For any graph G , $\chi(G) \leq \gamma_\ell(G)$.*

Both conjectures hold in the fractional setting. Reed proved that any graph satisfies $\chi_f(G) \leq \frac{1}{2}(\Delta(G) + 1 + \omega(G))$ [25]. McDiarmid observed that the proof could be modified to give a stronger result:

Theorem 1.3. *For any graph G , $\chi_f(G) \leq \max_{v \in V(G)} \frac{1}{2}(d(v) + 1 + \omega(v))$.*

The full proof appears in [20], §2.2. Thus we know that for any graph,

$$\chi_f(G) \leq \gamma_\ell(G) \leq \gamma(G) \leq \Delta(G) + 1.$$

Conjecture 1.1 was proved for line graphs by King, Reed, and Vetta [22]; we extended this to all quasi-line graphs [21]. Chudnovsky, King, Plumettaz and Seymour recently proved Conjecture 1.2 for line graphs [2]; the reductions from [21] also extend this result to all quasi-line graphs.

Theorem 1.4. *Given a quasi-line graph G , we can colour G using at most $\gamma_\ell(G)$ colours in polynomial time.*

Even more recently, Edwards and King proved that a stronger local version holds in the fractional setting and for quasi-line graphs [13], and conjectured that it always holds:

Conjecture 1.5 (Edwards and King). *For any graph G , $\chi(G) \leq \max_{uv \in E(G)} \lceil \frac{1}{2}(\gamma_\ell(u) + \gamma_\ell(v)) \rceil$.*

In this paper we prove that Conjecture 1.1 holds for all claw-free graphs, and Conjecture 1.2 holds for all claw-free graphs with a three-colourable complement, i.e. *three-cliqued* claw-free graphs:

Theorem 1.6. *For any claw-free graph G , $\chi(G) \leq \gamma(G)$.*

Theorem 1.7. *For any three-cliqued claw-free graph G , $\chi(G) \leq \gamma_\ell(G)$.*

Furthermore, both proofs yield polynomial-time algorithms. Theorem 1.6 complements a recent result of Chudnovsky and Seymour [8] for claw-free graphs with stability number $\alpha(G)$ at least three:

Theorem 1.8. *For any claw-free graph G with $\alpha(G) \geq 3$, $\chi(G) \leq 2\omega(G)$.*

Thus our result is stronger when $\Delta(G) + 1 \leq 3\omega(G)$ (in fact this is always the case when $\alpha(G) \geq 4$ or G is three-cliqued).

1.2 Overview

The structure theorem for claw-free graphs naturally divides our work into three types of claw-free graphs: those with a three-colourable complement, those constructed as a generalization of a line graph, and some remaining exceptional cases. Each of the first two categories involves some basic classes and a composition operation, such that every graph in that category is either basic or can be built from the basic graphs using the composition operation. Therefore our approach is to prove that Conjecture 1.2 holds for the basic classes, then prove that Conjecture 1.1 (and usually Conjecture 1.2) continues to hold when the composition operations are applied. Finally we deal with any remaining cases.

Before we do this, we introduce some machinery that allows us to simplify the class of graphs we need to consider. This is the notion of a *nonskeletal homogeneous pair of cliques*, or *NHPOC*. An NHPOC

can be thought of as a type of defect or “fuzziness”, and if one exists in a claw-free graph G , we can reduce to a proper claw-free subgraph G' without changing the chromatic number. Since γ and γ_ℓ are monotone graph invariants, a minimum counterexample to Theorem 1.6 or Theorem 1.7 cannot contain an NHPOC.

Nonskeletal (and other) homogeneous pairs of cliques are fundamental to the structure of claw-free graphs because of *thickenings*, a method of expanding vertices in claw-free graphs that generalizes the idea of *augmentations* introduced by Maffray and Reed [24]. In the next section we introduce thickenings and NHPOCs, and explain how we can restrict our focus to colouring *skeletal graphs*. Using skeletal graphs, we can easily prove that $\chi(G) \leq \gamma_\ell(G)$ for *antiprismatic thickenings*, an important class of claw-free graphs with $\alpha \leq 3$. These include all graphs with $\alpha \leq 2$, which are trivially claw-free. Thus we spend Section 2 introducing our tools and showing how to apply them effectively to some straightforward classes of claw-free graphs.

In Section 3 we present some important types of claw-free graphs that are fundamental to later constructions. In Section 4 we describe claw-free graphs with a three-colourable complement (*three-cliqued* claw-free graphs). They are built from several basic classes by a composition operation known as *hex-chains*. With both three-cliqued claw-free graphs and antiprismatic thickenings, our approach is to remove a stable set S for which $\gamma_\ell(G - S) < \gamma_\ell(G)$. This is not always possible; some types of three-cliqued graphs take a little more work. In Section 4 we complete the proof of Theorem 1.7, and then move on to proving Theorem 1.6.

To do this, we first need to deal with *compositions of strips*, whose structure generalizes that of line graphs and quasi-line graphs. In Section 5 we describe their structure and generalize our approach from [21]. In Section 6, we deal with the remaining case: the exceptional class of *icosahedral thickenings* (we deal with these after compositions of strips in order to introduce a certain decomposition where it is most sensible). This allows us to complete the proof of Theorem 1.6. Finally, in Section 7 we prove that our approach yields polynomial-time algorithms for constructing colourings that achieve our new bounds.

2 Skeletal graphs and thickenings

Chudnovsky and Seymour introduced thickenings, which generalize the operations of augmentation and multiplication, as a way to distill the essential structure of a graph or trigraph [7]. Here we describe thickenings and discuss how to reduce non-minimal structure that arises as a result of the thickening operation.

We *multiply* a vertex v by taking the disjoint union of $G - v$ and a nonempty clique $I(v)$, then making each vertex of $N(v)$ adjacent to each vertex of $I(v)$. In this case any two vertices of $I(v)$ are *twins*, i.e. they have the same closed neighbourhood. A clique C is a *homogeneous clique* if it has size between 2 and $n - 1$, and every vertex outside C sees either none or all of C . So as long as $I(v)$ is not a singleton or the entire graph, it is a homogeneous clique. Note that vertex multiplication will never introduce a claw when applied to a claw-free graph.

To generalize this operation, we consider edges whose deletion does not introduce a claw. We say that an edge e in a claw-free graph G is *claw-neutral* if $G - e$ is claw-free. A matching M is *claw-neutral* if every edge of M is claw-neutral. Observe that if M is claw-neutral, then $G - M$ is claw-free.

Let M be a claw-neutral matching in a claw-free graph G . We say that G' is a *thickening of G under M* (or sometimes just a thickening of G) if we can construct it from G in the following way. First we multiply each vertex. Then for every $uv \in M$, we remove from G' a nonempty proper subset of the edges between $I(u)$ and $I(v)$. If M is empty we say that G' is a *proper thickening* of G ; in this case G' simply arises from G by vertex multiplication. For a set $S \subseteq V(G)$ we use $I(S)$ to denote $\cup_{v \in S} I(v)$.

Just as proper thickenings give rise to homogeneous cliques, thickenings give rise to *homogeneous pairs of cliques*. A pair (A, B) of disjoint nonempty cliques is a homogeneous pair of cliques if $|A \cup B| \geq 3$ and every vertex outside $A \cup B$ sees all or none of A , and all or none of B . So for $u, v \in V(G)$, if $|I(u)| + |I(v)| \geq 3$ then $(I(u), I(v))$ is a homogeneous pair of cliques regardless of whether or not $uv \in E(G)$ or $uv \in M$.

It turns out that in a minimum counterexample to Theorem 1.6 or 1.7, we can guarantee that every homogeneous pair of cliques has a very simple structure. We address this issue now.

2.1 Skeletal graphs and skeletal homogeneous pairs

Given a homogeneous pair of cliques (A, B) in a graph G , we want to remove edges between A and B in G to reach a subgraph G' such that:

- G' is easier to describe and colour than G
- given a k -colouring of G' we can easily find a k -colouring of G .

In this paper we use two such reductions. A homogeneous pair of cliques (A, B) is *linear*¹ precisely if $G[A, B]$ contains no induced C_4 (equivalently, $G[A \cup B]$ is a *linear interval graph*, which we define later). Chudnovsky and Seymour used these to describe quasi-line graphs [4], and Chudnovsky and Fradkin used them to colour quasi-line graphs [3], as did we [21].

For claw-free graphs we need a stronger reduction. Observe that if we remove an edge between A and B without changing the chromatic number of the subgraph induced on $A \cup B$, the chromatic number of the graph will not change. Furthermore, since $G[A \cup B]$ is cobipartite and therefore perfect, $\chi(G[A \cup B]) = \omega(G[A \cup B])$. We say that (A, B) is *skeletal* if we cannot remove an edge between A and B without changing the clique number of $G[A \cup B]$. We say that G is *skeletal* if it contains no nonskeletal homogeneous pair of cliques. Observe that every skeletal homogeneous pair of cliques is linear.

Now for the reduction result. The following theorem immediately implies that a minimum counterexample to Theorem 1.6 or Theorem 1.7 must be skeletal.

Theorem 2.1. *Let G be a nonskeletal graph. Then there is a skeletal subgraph G' of G such that:*

1. *If G is quasi-line (resp. claw-free) then G' is also quasi-line (resp. claw-free).*
2. $\chi(G') = \chi(G)$ and $\chi_f(G') = \chi_f(G)$.
3. *If $\chi(\overline{G}) = 3$ then $\chi(\overline{G'}) = 3$.*

Furthermore we can find G' in $O(m(m^2 + n^{5/2}))$ time, and given a k -colouring of G' we can construct a k -colouring of G in $O(mn^{5/2})$ time.

This theorem follows immediately from at most m applications of the following two lemmas.

Lemma 2.2. *For any graph G , we can find a nonskeletal homogeneous pair of cliques, or determine that none exists, in $O(m^2)$ time.*

Lemma 2.3. *Given a graph G and a nonskeletal homogeneous pair of cliques (A, B) , in $O(n^{5/2})$ time we can remove edges between A and B to reach a proper subgraph G' such that:*

1. *(A, B) is a skeletal homogeneous pair of cliques in G' .*
2. *If G is quasi-line (resp. claw-free) then G' is also quasi-line (resp. claw-free).*
3. $\chi(G') = \chi(G)$ and $\chi_f(G') = \chi_f(G)$.
4. *If $\chi(\overline{G}) = 3$ then $\chi(\overline{G'}) = 3$.*

Furthermore given a k -colouring of G' we can construct a k -colouring of G in $O(n^{5/2})$ time.

Theorem 2.1 strengthens Lemma 9 from [21], which itself expands on Lemma 5.1 from [3]. We defer the proofs of Lemmas 2.2 and 2.3 to Section 8. If we only wanted to reduce nonlinear homogeneous pairs of cliques, we could use the faster and more sophisticated algorithm from [1].

¹These were originally called *nontrivial* homogeneous pairs of cliques by Chudnovsky and Seymour, who used them in their description of quasi-line graphs [4]. We prefer the more descriptive term *nonlinear* in part because they are less trivial than *skeletal* homogeneous pairs of cliques.

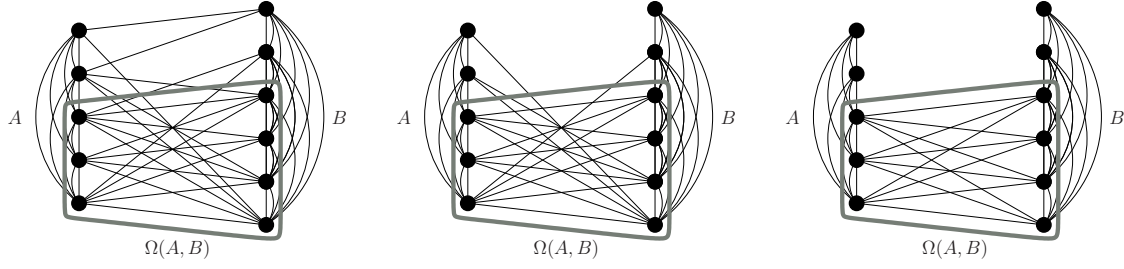


Figure 1: Three homogeneous pairs of cliques: one nonlinear (left), one nonskeletal linear (middle), and one skeletal (right). We reduce a nonskeletal homogeneous pair of cliques (A, B) by removing edges without changing the size of a maximum clique in $G[A \cup B]$.

2.1.1 The importance of being skeletal

If (A, B) is skeletal then the edges between A and B are contained in a single clique $\Omega(A, B)$, which we consider to be empty if there are no edges between A and B (see Figure 1). Thus $A \cup B$ can be partitioned into the four sets $A \cap \Omega(A, B)$, $B \cap \Omega(A, B)$, $A \setminus \Omega(A, B)$, $B \setminus \Omega(A, B)$, each of which is a homogeneous clique, a singleton, or empty. For convenience, when talking about a thickening we often use $\Omega(v_i, v_j)$ to denote $\Omega(I(v_i), I(v_j))$. We now explain why the structure of a skeletal homogeneous pair of cliques is so useful.

Our approach to colouring often involves removing a stable set S from a supposedly minimum counterexample G and confirming that for a given vertex set C , the removal of S causes $\max_{v \in C} (d(v) + \omega(v))$ to drop by two. We can easily insist that S be a maximal stable set, so $d(v) + \omega(v)$ drops by at least one for every vertex in $G - S$. In this case, removing S lowers $\max_{v \in C} \gamma_\ell(v)$. Thus we only need to worry about vertices in C maximizing $d(v) + \omega(v)$. In particular, if there is a vertex v in C whose closed neighbourhood properly contains the closed neighbourhood of another vertex v' , we can safely disregard v' in our analysis. In this case we say that v *trumps* v' .

Now consider the vertices in a skeletal homogeneous pair of cliques (A, B) . We can make several simple observations, all of which are symmetric with respect to A and B :

1. Every vertex in $A \setminus \Omega(A, B)$ is trumped by every vertex in $A \cap \Omega(A, B)$.
2. Removing a vertex from $A \cap \Omega(A, B)$ lowers $d(v)$ for any $v \in A \cup \Omega(A, B)$.
3. Removing a vertex from $A \cap \Omega(A, B)$ lowers $\omega(v)$ for any $v \in A$.
4. Removing a vertex from $A \cap \Omega(A, B)$ and a vertex from $B \setminus \Omega(A, B)$ lowers $d(v)$ by two for any $v \in B \cap \Omega(A, B)$, and lowers $\omega(v)$ for any $v \in B \setminus \Omega(A, B)$. In particular, it lowers $\max_{v \in A \cup B} (d(v) + \omega(v))$ by two.

We now prove that Theorem 1.7 holds for *antiprismatic thickenings* by exploiting the simplicity of skeletal homogeneous pairs of cliques.

2.2 Antiprismatic thickenings

A *triad* is a stable set of size three. A graph G is *antiprismatic* if every triad T contains exactly two neighbours of every vertex in $G - T$. Such graphs are clearly claw-free, and they were described in detail by Chudnovsky and Seymour [5, 6]. We say that an edge $e = uv$ in an antiprismatic graph G is *changeable* if $G - e$ is also antiprismatic. If this is the case, then (i) in G , neither u nor v is in a triad, and (ii) in $G - e$, u and v are in at most one triad (see [5], §16).

Given a matching M , we say that M is a *changeable matching* in G if for every $M' \subseteq M$, $G - M'$ is antiprismatic. If M is a changeable matching in G , then M is claw-neutral in G . If G' is a thickening

of an antiprismatic graph G under a changeable matching M , then we say that G' is an *antiprismatic thickening*. In this section we prove that $\chi \leq \gamma_\ell$ for antiprismatic thickenings.

2.2.1 The case $\alpha \leq 2$

We begin with trivially antiprismatic graphs, i.e. graphs containing no triad. In these graphs, a colouring corresponds to a matching in the complement, and we can therefore appeal to well-known results in matching theory.

Theorem 2.4. *Let G be any graph with $\alpha(G) \leq 2$. Then $\chi(G) \leq \gamma_\ell(G)$.*

Our proof relies on the observation that an optimal colouring of a graph with $\alpha \leq 2$ corresponds to a maximum matching in the complement \overline{G} . Rabern [29] independently proved that $\chi \leq \gamma$ for such graphs using a similar approach.

Proof of Theorem 2.4. Let G be a minimum counterexample to the theorem. Applying the Edmonds-Gallai structure theorem ([12, 17], see also [20] §2.5) for maximum matchings tells us that either (i) there is a vertex $v \in G$ such that $\chi(G) = \chi(G - v)$, (ii) \overline{G} is not connected, or (iii) \overline{G} has a matching of size $\lfloor \frac{n}{2} \rfloor$ and consequently $\chi(G) = \lceil \frac{n}{2} \rceil$. Minimality of G tells us that (i) is impossible.

Suppose \overline{G} is not connected. Then $V(G)$ can be partitioned into nonempty V_1 and V_2 such that V_1 is joined to V_2 , i.e. every possible edge between V_1 and V_2 exists. It is easy to confirm that $\chi(G) = \chi(G[V_1]) + \chi(G[V_2]) \leq \gamma_\ell(G[V_1]) + \gamma_\ell(G[V_2]) \leq \gamma_\ell(G)$, the middle inequality following from the minimality of G .

Therefore (iii) must be the case, so $\chi(G) = \lceil \frac{n}{2} \rceil$. Since $\chi_f(G) \geq \frac{n}{\alpha(G)}$, we have $\chi(G) = \lceil \chi_f(G) \rceil$. By Theorem 1.3,

$$\chi(G) \leq \lceil \chi_f(G) \rceil \leq \max_{v \in V(G)} \left\lceil \frac{1}{2}(d(v) + 1 + \omega(v)) \right\rceil.$$

This proves the theorem. \square

It is not hard to prove the case $\chi(G) = \lceil \frac{n}{2} \rceil$ without using Theorem 1.3. However, this application of Theorem 1.3 is a useful trick and we will use it again later in the paper.

2.2.2 The case $\alpha = 3$

It remains to show that $\chi(G) \leq \gamma_\ell(G)$ for any antiprismatic thickening G containing a triad. This case is fairly easy, and is a perfect example of a method we will use repeatedly: Given a supposed minimum counterexample G , we remove a stable set T (in this case a triad) such that $\gamma_\ell(G - T) < \gamma_\ell(G)$. This immediately contradicts the minimality of our supposed counterexample, since we can make the triad T a colour class in a $\chi(G - T) + 1$ colouring of G . We first define the type of triad we seek; we will use them repeatedly. Recall from Section 2.1.1 that a vertex u *trumps* a vertex v if $\tilde{N}(v) \subset \tilde{N}(u)$.

Definition. *Let T be a triad in a graph G . If every vertex v in $G - T$ has two neighbours in T or a twin in T or is trumped by a vertex in T , then we say that T is a good triad.*

Observe that any good triad T has the property that $\gamma_\ell(G - T) \leq \gamma_\ell(G) - 1$.

Theorem 2.5. *Let G be an antiprismatic thickening. Then $\chi(G) \leq \gamma_\ell(G)$.*

Proof. Let G be a minimum counterexample to the theorem. We already know that $\alpha(G) = 3$. If G contains a good triad T , then since $\chi(G - T) \leq \gamma_\ell(G - T)$ and $\chi(G - T) \geq \chi(G) - 1$, we know that $\chi(G) \leq \gamma_\ell(G)$. Therefore to reach a contradiction it suffices to prove the existence of a good triad. Suppose G is a thickening of an antiprismatic graph H under a changeable matching M .

Suppose there is a triad $\{u, v, w\}$ in H . Then note that by the properties of a changeable edge, none of u, v, w is an endpoint of any edge e in M : the other endpoint y would either form a claw with T , or y would have only one neighbour in T in $G - e$, contradicting the fact that M is changeable. Let T be a

triad in $I(u) \cup I(v) \cup I(w)$. Every vertex in $(I(u) \cup I(v) \cup I(w)) \setminus T$ has a twin in T , and every vertex in $G - (I(u) \cup I(v) \cup I(w))$ has two neighbours in T . Therefore T is a good triad and we are done.

So there is no triad in H . Since $\alpha(G) = 3$, there are vertices u, v, w in H such that $e = uv \in M$ and $\{u, v, w\}$ is a triad in $H - e$. By the definition of a thickening, $I(u) \cup I(v)$ is not a clique but there is at least one edge between $I(u)$ and $I(v)$.

We claim that $(I(u), I(v))$ is a skeletal homogeneous pair of cliques in G . For if this is not the case, Lemma 2.3 tells us that we can remove edges between $I(u)$ and $I(v)$ to reach a proper subgraph G' of G with $\chi(G') = \chi(G)$; one can easily confirm that G' is either a thickening of H under M , or a thickening of $H - e$ under $M - e$. Either way, G' is an antiprismatic thickening and contradicts the minimality of G . Therefore $(I(u), I(v))$ is skeletal, $\Omega(u, v)$ is nonempty, and at least one of $I(u) \setminus \Omega(u, v)$ and $I(v) \setminus \Omega(u, v)$ is nonempty. Assume $I(u) \setminus \Omega(u, v)$ is nonempty. Let $a, b, c \in V(G)$ be vertices in $I(u) \setminus \Omega(u, v)$, $I(v) \cap \Omega(u, v)$, and $I(w)$ respectively, and note that $T = \{a, b, c\}$ is a triad. It suffices to show that it is a good triad, which we do now.

Observe that w is not in $V(M)$, for if there were an edge $wx \in M$ then since $H - e$ is antiprismatic, x would have two neighbours in $\{u, v, w\}$ in $H - e$, contradicting the fact that $H - e - wx$ must also be antiprismatic since M is a changeable matching in H . Since $H - e$ is antiprismatic, any vertex of G without two neighbours in T must be in $I(u) \cup I(v) \cup I(w)$. Therefore a vertex in $I(w) \setminus T$ has a twin in T , a vertex in $I(u) \setminus T$ has two neighbours or a twin in T (depending on whether or not it is in $\Omega(u, v)$), and a vertex in $I(v) \setminus T$ has a twin in T or is trumped by a vertex in T (again depending on whether or not it is in $\Omega(u, v)$). Therefore T is a good triad and we are done. \square

The proof actually implies a slightly different result, which is worth stating separately:

Corollary 2.6. *Let G be a skeletal antiprismatic thickening with $\alpha(G) \geq 3$. Then G contains a good triad.*

In Section 7 we will show that given an antiprismatic thickening G of an antiprismatic graph H under a changeable matching M , we can find H and M in polynomial time.

3 Some important types of claw-free graphs

To fully describe skeletal claw-free graphs we must first define some fundamental subclasses, the first of which was antiprismatic thickenings. Here we describe line graphs, linear and circular interval graphs, and antihat thickenings.

3.1 Line graphs

Given a multigraph H , its *line graph* $L(H)$ is the graph with one vertex for each edge of H , in which two vertices are adjacent precisely if their corresponding edges in H share at least one endpoint. We say that G is a line graph if $G = L(H)$ for some multigraph H . Thus the neighbours of any vertex v in a line graph $L(H)$ are covered by two cliques, one for each endpoint of the edge in H corresponding to v . Observe that every line graph is claw-free. When considering the line graph of H we may assume that H is loopless, since replacing a loop with a pendant edge in H will not change $L(H)$.

Suppose G is the line graph of H , and that G contains a matching M in which each edge corresponds to the two edges in H incident to some vertex of degree 2. Then M is a claw-neutral matching, and any thickening of G under M is a thickening what Chudnovsky and Seymour call a *thickening of a line trigraph* [7]. Now suppose G' is a skeletal thickening of G under M . We claim that G' is actually a line graph as well:

Proposition 1. *If a graph G' is a thickening of a line trigraph and is skeletal, then G' is a line graph.*

Proof. Let G' be a skeletal thickening of a line graph G under a matching M as described in the paragraph above. Consider an edge $uv \in M$ and the corresponding homogeneous pair of cliques $(I(u), I(v))$ in G' .

Every vertex in $(I(u) \cup I(v)) \setminus \Omega(u, v)$ is simplicial. Therefore G' is a thickening of a line graph $L(H')$ under a matching $M \setminus \{uv\}$, where H' is constructed from H looking at the unshared endpoints of u and v and adding a pendant edge to each. Repeating this process for each edge in M proves the claim. \square

It is useful to bear this fact in mind when we define the class \mathcal{TTC}_1 in Section 4.

3.2 Linear interval graphs, circular interval graphs, and quasi-line graphs

One class of graphs lying between line graphs and claw-free graphs is the class of *quasi-line graphs*. A graph is quasi-line if the neighbourhood of every vertex induces the complement of a bipartite graph. We now present two fundamental types of quasi-line graphs.

A *linear interval graph* is a graph $G = (V, E)$ with a *linear interval representation*, which is a point on the real line for each vertex and a set of intervals such that vertices u and v are adjacent in G precisely if there is an interval containing both corresponding points on the real line. Linear interval graphs are chordal and therefore perfect.

In the same vein, a *circular interval graph* is a graph with a *circular interval representation*, which consists of $|V|$ points on the unit circle and a set of intervals (arcs) on the unit circle such that two vertices of G are adjacent precisely if some arc contains both corresponding points. This class contains all linear interval graphs. Deng, Hell, and Huang proved that we can identify and find a representation of a circular or linear interval graph in linear time [11].

A circular interval graph is a *long circular interval graph* if it has a circular interval representation in which no three intervals cover the entire circle. Note that it is still possible for three intervals to cover all vertices.

Theorem 1.4 tells us that every quasi-line graph satisfies $\chi(G) \leq \gamma_\ell(G)$. For circular interval graphs, this bound follows easily from known results. First, Niessen and Kind [26] proved that circular interval graphs have the *round-up property*:

Lemma 3.1. *For any circular interval graph G , $\chi(G) = \lceil \chi_f(G) \rceil$.*

A result of Shih and Hsu [31] tells us that we can optimally colour circular interval graphs efficiently:

Lemma 3.2. *Given a circular interval graph G , we can find an optimal colouring of G in $O(n^{3/2})$ time.*

These results, along with Theorem 1.3, immediately imply that Theorem 1.7 holds for circular interval graphs.

Lemma 3.3. *If G is a circular interval graph, we can find a $\gamma_\ell(G)$ -colouring of G in polynomial time.*

3.3 Antihat thickenings

We need to consider certain thickenings of graphs that are nearly antiprismatic. Let $k \geq 2$. We first define a graph H with vertex set $A \cup B \cup C$ as follows. Let $A = \{a_0, a_1, \dots, a_k\}$, $B = \{b_0, b_1, \dots, b_k\}$, and $C = \{c_1, \dots, c_k\}$ be disjoint cliques. Adjacency between the cliques is as follows:

- a_0 has no neighbour outside $A \cup \{b_0\}$, and b_0 has no neighbour outside $B \cup \{a_0\}$.
- For $1 \leq i, j \leq k$, a_i and b_j are nonadjacent if $i \neq j$ and adjacent if $i = j$.
- For $1 \leq i, j \leq k$, a_i and b_i are adjacent to c_j if $i \neq j$, and nonadjacent to c_j if $i = 0$ or if $i = j$.

Let $X \subset A \cup B \cup C \setminus \{a_0, b_0\}$ such that $|C \setminus X| \geq 2$, and let $G = H - X$. We say that G is an *antihat graph*. To define antihat thickenings, we first define a set $M \in V(G)^2$ as follows:

- M is a matching in $G \cup M$ containing no edge of $G[A]$, $G[B]$, or $G[C]$.
- $a_0 b_0$ is in M if a_0 and b_0 are adjacent in G .

- If $1 \leq i, j$ and $a_i b_j \in M$ then $i = j$ and $c_i \in X$.
- If $1 \leq i, j$ and $b_i c_j \in M$ then $i = j$ and $a_i \in X$.
- If $1 \leq i, j$ and $a_i c_j \in M$ then $i = j$ and $b_i \in X$.

In this case $G \cup M$ is claw-free and M is a claw-neutral matching in $G \cup M$. If G' is a thickening of $G \cup M$ under M then we say that it is an *antihat thickening*. Observe that given an antihat graph G , adding an edge between a_0 and b_0 gives us an antiprismatic graph, as does deleting one or both of a_0 and b_0 .

Having presented these graph classes, we can move on to the next step: describing and colouring three-cliqued claw-free graphs.

4 Three-cliqued claw-free graphs

We now consider claw-free graphs with a three-colourable complement. Given cliques A , B , and C that partition the vertices of a claw-free graph G , we say that (G, A, B, C) is a *three-cliqued claw-free graph*. We also sometimes just call G a three-cliqued claw-free graph without specifying a 3-colouring of \overline{G} . As we will state formally in Theorem 4.1, any skeletal three-cliqued claw-free graph either admits a *hex-join*, which we describe shortly, or belongs to one of several base classes.

4.1 Base classes of three-cliqued claw-free graphs

Since we restrict our attention to skeletal claw-free graphs, we can restrict the base classes of hex-joins that we describe. However, it is possible to compose two nonskeletal three-cliqued claw-free graphs with a hex-join and reach a skeletal graph, so we cannot assume the base graphs are skeletal. We therefore consider *weakly skeletal* base graphs, i.e. those in which every nonskeletal homogeneous pair of cliques has one clique intersecting at least two of A , B , and C :

Definition. Let (X, Y) be a homogeneous pair of cliques in a three-cliqued graph (G, A, B, C) . Then (X, Y) is *straddling* if at least one of X or Y intersects more than one of A , B , and C . We say that (G, A, B, C) is *weakly skeletal* if every nonskeletal homogeneous pair of cliques is straddling.

The first four classes we define contain weakly skeletal thickenings of members of the classes $\mathcal{TC}_1, \dots, \mathcal{TC}_4$ as defined by Chudnovsky and Seymour [7].

- **A type of line graph.** Let H be a multigraph with pairwise nonadjacent vertices a, b, c such that each of a, b, c has at least three neighbours, and such that every edge of H has an endpoint in $\{a, b, c\}$. We further insist that for each $S \subset \{a, b, c\}$ there is at most one vertex u outside $\{a, b, c\}$ whose neighbourhood is S . Let $G = L(H)$, and let cliques A , B , and C in G correspond to the edges incident to a , b , and c respectively in H . Then (G, A, B, C) is a three-cliqued claw-free graph. Let \mathcal{TTTC}_1 be the set of all such three-cliqued graphs such that every vertex is in a triad, with the added condition of being weakly skeletal.²
- **Long circular interval graphs.** Let (G, A, B, C) be a three-cliqued long circular interval graph with a circular interval representation such that each of A , B , C is a set of consecutive vertices in circular order. Let \mathcal{TTTC}_2 be the set of all such graphs that are weakly skeletal, such that every vertex is in a triad.
- **Antihat thickenings.** Let G be an antihat thickening, and let A, B, C , and X be as they are in the definition of G . Let $A' = A \setminus X$ and define B' and C' accordingly. Then $(G - I(X), I(A'), I(B'), I(C'))$ is a three-cliqued claw-free graph. Let \mathcal{TTTC}_3 be the class of all such three-cliqued graphs with the added condition of being weakly skeletal.

²To see that these graphs correspond to weakly skeletal thickenings of trigraphs in \mathcal{TC}_1 from [7], recall Proposition 1 and its proof.

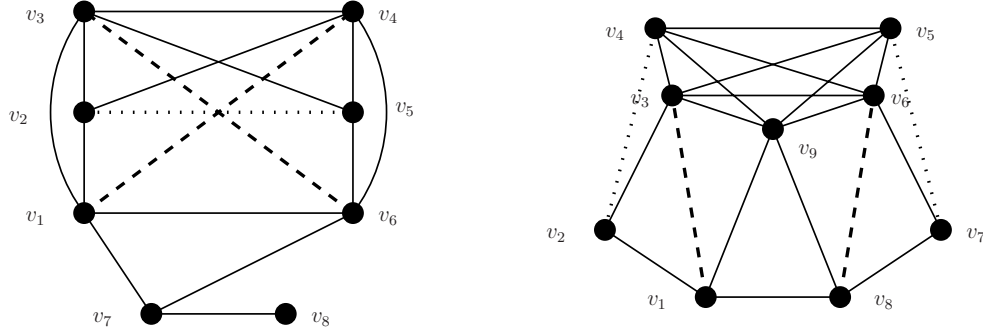


Figure 2: The graphs underlying exceptional thickenings in \mathcal{TC}'_5 (left) and \mathcal{TC}''_5 (right). Solid, dashed, and dotted lines represent adjacent vertices, edges in M , and unspecified adjacency respectively. All other pairs are nonadjacent.

- **Antiprismatic thickenings.** Let (G, A, B, C) be a three-cliqued antiprismatic graph, and let $(G', I(A), I(B), I(C))$ be a thickening of G under a changeable matching M . Let \mathcal{TTC}_4 be the class of all such graphs $(G', I(A), I(B), I(C))$ that are weakly skeletal.

The final two exceptional cases correspond to thickenings of graphs in Chudnovsky and Seymour's class \mathcal{TC}_5 [7].

- **Exception I.** Let G be a graph on vertices v_1, \dots, v_8 with adjacency as follows: v_1 is adjacent to v_2, v_3, v_6, v_7 ; v_2 is adjacent to v_3, v_4 ; v_3 is adjacent to v_4, v_5 ; v_4 is adjacent to v_5, v_6 ; v_5 is adjacent to v_6 ; v_6 and v_8 are adjacent to v_7 ; v_2 may or may not be adjacent to v_5 . There are no other edges. Now let M be a matching containing v_1v_4 , v_3v_6 , and also v_2v_5 if $v_2v_5 \in E(G)$. Let $X \subseteq \{v_3, v_4\}$. Let G' be a thickening of $(G \cup M) - X$ under M (see Figure 2). Then $(G', I(\{v_1, v_2, v_3\}), I(\{v_4, v_5, v_6\}), I(\{v_7, v_8\}))$ is a three-cliqued claw-free graph. Let \mathcal{TTC}_5 be the set of all such graphs with the added condition of being weakly skeletal.
- **Exception II.** Let G be a graph on vertices v_1, \dots, v_9 with the following structure. Let $A = \{v_1, v_2\}$, $B = \{v_7, v_8\}$, and $C = \{v_3, v_4, v_5, v_6, v_9\}$ be cliques. Let v_1 be adjacent to v_3, v_8 , and v_9 . Let v_8 be adjacent to v_6 and v_9 . Let v_2 be adjacent to v_3 and possibly v_4 . Let v_7 be adjacent to v_6 and possibly v_5 . Now let M be a matching in G containing v_1v_3 and v_6v_8 , as well as possibly v_2v_4 and v_5v_7 (see Figure 2). Let X be a subset of $\{v_3, v_4, v_5, v_6\}$ such that:
 - v_2 and v_7 each have a neighbour in $C \setminus X$.
 - If X contains neither v_4 nor v_5 then v_2 is adjacent to v_4 and v_7 is adjacent to v_5 .

We insist that every vertex of $(G - M) - X$ is in a triad. Let G' be a thickening of $G - X$ under M . Then $(G', I(A), I(B), I(C \setminus X))$ is a three-cliqued claw-free graph. Let \mathcal{TTC}_6 be the set of all such graphs with the added condition of being weakly skeletal.

We allow permutations of the sets A, B, C for any of these classes, so if (G, A, B, C) is in \mathcal{TTC}_i for some $1 \leq i \leq 6$ and $\{A', B', C'\} = \{A, B, C\}$, then (G, A', B', C') is also in \mathcal{TTC}_i . Having described the building blocks for three-cliqued claw-free graphs, we now move on to how they are combined (or from our perspective, decomposed).

4.2 Decomposition: hex-joins

We can decompose skeletal three-cliqued claw-free graphs into the base classes we just defined using a single decomposition operation: hex-joins. Let (G, A, B, C) be a three-cliqued graph, and suppose we partition A into A_1, A_2 , B into B_1, B_2 , C into C_1, C_2 . Let $G_1 = G[A_1 \cup B_1 \cup C_1]$ and let $G_2 =$

$G[A_2 \cup B_2 \cup C_2]$. Suppose we can construct G from the disjoint union of G_1 and G_2 by adding every possible edge between A_1 and A_2 , A_2 and B_1 , B_1 and B_2 , B_2 and C_1 , C_1 and C_2 , and C_2 and A_1 . Then we say that (G, A, B, C) admits a *hex-join* into (G_1, A_1, B_1, C_1) and (G_2, A_2, B_2, C_2) .

A simple observation explains our focus on weakly skeletal base graphs:

Observation 1. *Let (X, Y) be a nonskeletal, non-straddling homogeneous pair of cliques in a three-cliqued graph (G_1, A_1, B_1, C_1) . If (G, A, B, C) admits a hex-join into (G_1, A_1, B_1, C_1) and any three-cliqued graph (G_2, A_2, B_2, C_2) , then (X, Y) is a nonskeletal homogeneous pair of cliques in (G, A, B, C) . In particular, (G, A, B, C) is not skeletal.*

We use the following decomposition theorem for skeletal three-cliqued claw-free graphs. It is a straightforward weakening of Chudnovsky and Seymour's structure theorem for three-cliqued claw-free trigraphs (4.1 in [7]), as discussed in Chapter 9 of [20].

Theorem 4.1. *Any skeletal three-cliqued claw-free graph (G, A, B, C) not in \mathcal{TTC}_4 admits a hex-join into terms (G_1, A_1, B_1, C_1) and (G_2, A_2, B_2, C_2) , where (G_1, A_1, B_1, C_1) is in one of \mathcal{TTC}_1 , \mathcal{TTC}_2 , \mathcal{TTC}_3 , \mathcal{TTC}_5 , or \mathcal{TTC}_6 .*

The omission of \mathcal{TTC}_4 from the list of possibilities comes from the easy fact that a graph admitting a hex-join into two terms, both of which are in \mathcal{TTC}_4 , will itself be in \mathcal{TTC}_4 .

Remark: The reader familiar with the structure of *claw-free trigraphs* may object to our omission of *worn hex-joins*, described in [7]. This omission is possible because if (G, A, B, C) admits a worn hex-join into (G_1, A_1, B_1, C_1) and (G_2, A_2, B_2, C_2) , where (G_1, A_1, B_1, C_1) is in one of \mathcal{TTC}_1 , \mathcal{TTC}_2 , \mathcal{TTC}_3 , \mathcal{TTC}_5 , or \mathcal{TTC}_6 , then that worn hex-join is actually a hex-join, since every vertex in one of these classes arises as the image, in a thickening, of a vertex that is in a triad in the *trigraph* sense.

4.3 Colouring three-cliqued claw-free graphs

We now prove our first main result, Theorem 1.7, which states that every three-cliqued claw-free graph G satisfies $\chi(G) \leq \gamma_\ell(G)$.

To bound the chromatic number of antiprismatic thickenings, we removed a good triad whenever possible. We will do the same for the remaining types of three-cliqued claw-free graphs. A claw-free graph containing no triad is necessarily antiprismatic, but not all three-cliqued claw-free graphs containing a triad contain a good triad. Observe that no minimum counterexample to Theorem 1.7 contains a good triad. Furthermore, good triads behave nicely with respect to hex-joins:

Observation 2. *Suppose that a three-cliqued claw-free graph (G, A, B, C) admits a hex-join into (G_1, A_1, B_1, C_1) and (G_2, A_2, B_2, C_2) . If T is a good triad in G_1 , then it is also a good triad in G .*

Let G be a minimum counterexample to Theorem 1.7. Then G is skeletal and is not an antiprismatic thickening. So Theorem 4.1 implies that G admits a hex-join into (G_1, A_1, B_1, C_1) and (G_2, A_2, B_2, C_2) , where (G_1, A_1, B_1, C_1) is in \mathcal{TTC}_1 , \mathcal{TTC}_2 , \mathcal{TTC}_3 , \mathcal{TTC}_5 , or \mathcal{TTC}_6 . We deal with these five possibilities individually. Note that G_2 may be empty, but this does not affect our approach.

4.3.1 Five classes to consider

We now prove a set of lemmas that together imply Theorem 1.7, dealing with the easier cases first.

Long circular interval graphs (\mathcal{TTC}_2)

Lemma 4.2. *Any three-cliqued graph (G_1, A_1, B_1, C_1) in \mathcal{TTC}_2 contains a good triad.*

Proof. Suppose that (G_1, A_1, B_1, C_1) is in \mathcal{TTC}_2 , and call the vertices of G $\{a_1, \dots, a_i, b_1, \dots, b_j, c_1, \dots, c_k\}$ in circular order.

We can find a triad T containing a_1 by adding b_p for the minimum p such that b_p does not see a_1 , then adding c_q for the minimum q such that b_p does not see c_q . The triad T exists since a_1 is in a triad, and it follows from the structure of circular interval graphs that a_1 and b_p are in a triad together. If some vertex in $(A \setminus \{a_1\}) \cup \{b_x \mid x < p\}$ does not see both a_1 and b_p , then we are in a degenerate case where G_1 is a linear interval graph, and the vertex in question is a twin of a_1 or b_p , or it is trumped by a_1 or b_p . The same applies to every vertex in $\{b_x \mid x > p\} \cup \{c_y \mid y < q\}$: each vertex has two neighbours in T or a twin in T or is trumped by a vertex in T . Similarly, if some vertex v in $\{c_l \mid l > q\}$ has only one neighbour in T then it has no neighbours in A , hence it is trumped by or is a twin of c_q . Thus T is a good triad. \square

Antihat thickenings (\mathcal{TTC}_3)

Lemma 4.3. *Any three-cliqued graph (G_1, A_1, B_1, C_1) in \mathcal{TTC}_3 contains a good triad.*

Proof. Let T be a triad consisting of a vertex a of $I(a_0)$ and vertices b in $I(B \setminus \{b_0\})$ and c in $I(C)$ respectively, following the definition of an antihat thickening. If b and c are in $I(b_i)$ and $I(c_i)$ respectively, we insist that T intersects $\Omega(b_i, c_i)$ if it is not empty. We also insist that if $I(a_0) \cap \Omega(a_0 b_0)$ is nonempty, then T intersects it. It is easy to confirm from the structure of an antihat thickening that T exists and is a good triad. \square

Exception I (\mathcal{TTC}_5)

Lemma 4.4. *Any three-cliqued graph (G_1, A_1, B_1, C_1) in \mathcal{TTC}_5 contains a good triad.*

Proof. Let T be a triad including one vertex in each of $I(v_7)$, $I(v_2)$, and $I(v_5)$, such that T intersects $\Omega(v_2 v_5)$ if it is not empty. It is easy to confirm that T is a good triad: Vertices in $I(\{v_1, v_3, v_4, v_6\})$ have two neighbours in T , and vertices in $I(\{v_7, v_8\})$ have a neighbour or a twin in T . If $\Omega(v_2 v_5)$ is empty then vertices in $I(\{v_2, v_5\})$ have a twin in T . If not, then assume without loss of generality that T intersects $I(v_2) \cap \Omega(v_2 v_5)$ and $I(v_5) \setminus \Omega(v_2 v_5)$. Then vertices in $I(v_2) \cap \Omega(v_2 v_5)$ have a twin in T , vertices in $I(v_2) \setminus \Omega(v_2 v_5)$ are trumped by a vertex in T , vertices in $I(v_5) \cap \Omega(v_2 v_5)$ have two neighbours in T , and vertices in $I(v_5) \setminus \Omega(v_2 v_5)$ have a twin in T . Therefore T is a good triad. \square

Exception II (\mathcal{TTC}_6)

Lemma 4.5. *Any three-cliqued graph (G_1, A_1, B_1, C_1) in \mathcal{TTC}_6 contains a good triad.*

Proof. Let T be a triad including one vertex in each of $I(v_2)$, $I(v_7)$, and $I(v_9)$, such that T intersects $\Omega(v_2 v_4)$ if it is not empty, and intersects $\Omega(v_5 v_7)$ if it is not empty. It is easy to confirm that T is a good triad (see Figure 2). \square

A type of line graph (\mathcal{TTC}_1)

We now prove the necessary lemma for G_1 in \mathcal{TTC}_1 . This is by far the most difficult case. We make extensive use of the fact that line graphs of bipartite multigraphs are perfect.

Lemma 4.6. *Let (G, A, B, C) be a minimum counterexample to Theorem 1.7 and suppose it admits a hex-join into (G_1, A_1, B_1, C_1) and (G_2, A_2, B_2, C_2) . Then (G_1, A_1, B_1, C_1) is not in \mathcal{TTC}_1 .*

Proof. Suppose (G_1, A_1, B_1, C_1) is in \mathcal{TTC}_1 . Then G_1 is the line graph of some bipartite multigraph H which has a stable set $\{a, b, c\}$ corresponding to A_1 , B_1 , and C_1 . Assume without loss of generality that $|C_1| \leq |B_1| \leq |A_1|$. We call the other vertices of H *centres*. Depending on the structure of G_1 we will take one of two actions:

1. Remove a triad from G_1 , lowering $\gamma_\ell(G)$.
2. Remove edges from G_1 without changing $\chi(G)$ or changing the fact that (G_1, A_1, B_1, C_1) is three-cliqued or claw-free.

Every vertex of G_1 is in a triad. If there are only three centres then removing any triad T will lower $\gamma_\ell(G)$ since every vertex in $G_1 - T$ will have two neighbours or a twin in T – this can be confirmed easily since the graph underlying H will be a subgraph of $K_{3,3}$. So there are at least four centres. Call the four centres of highest degree w, x, y , and z such that $d(w) \geq d(x) \geq d(y) \geq d(z)$.

For any centre s , denote by A_s the clique corresponding to the edges of H between a and s . Define B_s and C_s accordingly. Denote $A_s \cup B_s \cup C_s$ by X_s . We now consider, for some vertex $v \in A_s$, what cliques of size $\omega(v)$ can contain v . By the structure of a hex-join, observe that such a clique must be one of:

- A clique in G_1 intersecting all of A_1, B_1, C_1 . Specifically, $A_s \cup B_s \cup C_s = X_s$.
- A clique in $A_1 \cup B_1 \cup A_2$ containing all of A_2 . Specifically, $A_2 \cup A_s \cup B_s$.
- A clique in $A_1 \cup C_1 \cup C_2$ containing all of C_2 . Specifically, $C_2 \cup A_s \cup C_s$.
- A clique in $A_1 \cup A_2 \cup C_2$ containing all of A_1 . Such a clique has size at least $|A_1| + \max\{|A_2|, |C_2|\}$.

Note also that the closed neighbourhood of v is $A_1 \cup X_s \cup A_2 \cup C_2$. We can make similar observations about the cliques of size $\omega(v)$ when v is in $A_1 \setminus A_s$ or B_1 or C_1 . These observations help us characterize the situations in which removing a triad T lowers $\omega(v)$ and therefore $\gamma_\ell(v)$.

Note that at most two centres have degree $\geq d(a)$, since there are at least four centres and the sum of their degrees is $d(a) + d(b) + d(c)$. Suppose there are at most three centres with degree $\geq d(c)$. Then the structure of \mathcal{TTC}_1 tells us that we can find a matching of size 3 in H hitting each of these centres having degree greater than 1. We will now show that removing the corresponding triad T from G will lower $\gamma_\ell(v)$ for all $v \in G_1$. This triad T will hit A_1, B_1 , and C_1 . Any vertex v in G_1 without two neighbours or a twin in T , that is not trumped by a vertex in T , will correspond to an edge in H incident to some centre s , where $d(s) < d(c)$. By our above observations about cliques of size $\omega(v)$, we can see that since $|X_s| < |C_1| \leq |B_1| \leq |A_1|$, any clique of size $\omega(v)$ containing v must contain one of C_1, B_1 , or A_1 . Therefore such a clique intersects T , so removing T lowers $\omega(v)$ and also $\gamma_\ell(v)$. This contradicts the minimality of G , so we can assume that there are at least four centres of degree $\geq |C|$, i.e. $d(z) \geq d(c)$. Given this restriction we now consider several cases.

Case 1: $d(w) \geq d(a)$ and c sees w .

Since $d(w) \geq d(a)$ it follows that $d(x) + d(y) + d(z) \leq |B_1| + |C_1|$, and so $d(x) + d(y) \leq |B_1|$ and $|C_1| \leq d(z) \leq \frac{1}{3}(|B_1| + |C_1|)$. Therefore $2|C_1| \leq 2d(z) \leq d(x) + d(y) \leq |B_1|$. Take a triad T that hits X_w, X_x , and X_y , and consider a vertex v for which $\omega(v)$ does not drop when T is removed. Clearly v is not in $X_w \cup X_x \cup X_y$, so it is in X_s for some centre s with $d(s) \leq d(z) \leq \frac{1}{2}|B_1|$. Since $|X_s| < |B_1|$ and $\omega(v)$ does not drop, v must be in C_s . Take some $u \in C_w$. We will show that $d(u) + \omega(u) > d(v) + \omega(v)$, which implies that $\gamma_\ell(G - T) < \gamma_\ell(T)$.

Clearly u has at least $|A_1| - |C_1|$ neighbours in $G_1 - C_1$. But v has at most $\frac{1}{2}|B_1| - 1$ neighbours in $G_1 - C_1$. Therefore $d(u) > d(v) + \frac{1}{2}|A_1| - |C_1|$. Recall the structure of maximal cliques containing u and v . If $\omega(v) > \omega(u)$ then either $|C_s| + |A_s| > \max\{|C_w| + |A_w|, |C_1|\}$ or $|C_s| + |B_s| > \max\{|C_w| + |B_w|, |C_1|\}$. But in this case $\omega(v) \leq \omega(u) + d(s) - |C_1| \leq \omega(u) + \frac{1}{2}|A_1| - |C_1|$. It follows that $d(v) + \omega(v) < d(u) + \omega(u)$, completing the case.

Case 2: $d(w) \geq d(a)$ and c does not see w .

Make the subgraph G' of G by removing all edges between C_1 and $G_1 \setminus C_1$ – observe that (G', A, B, C) is claw-free and three-cliqued. Further observe that because H has at least four centres, if $G' = G$ then (A_1, B_1) is a nonskeletal homogeneous pair of cliques in both G_1 and G , a contradiction. Thus G' is a proper subgraph of G . We claim that $\chi(G') = \chi(G)$, contradicting the minimality of G . Denote by G'_1 the subgraph of G' induced on the vertices of G_1 .

Take a $\chi(G')$ -colouring \mathcal{C}' of G' . We will rearrange the colour classes of \mathcal{C}' on G'_1 to reach a proper colouring of G_1 . Denote by t the number of triad colour classes in \mathcal{C}' restricted to G'_1 . Denote by d_{AB} , d_{AC} , and d_{BC} the number of diads (i.e. colour classes of size two) in \mathcal{C}' restricted to G'_1 intersecting A_1 and B_1 , A_1 and C_1 , and B_1 and C_1 respectively. It suffices to show that we can pack the appropriate disjoint stable sets into G_1 . That is, we want to find t triads in G_1 , d_{AB} diads intersecting A_1 and B_1 , d_{AC} diads intersecting A_1 and C_1 , and d_{BC} diads intersecting B_1 and C_1 , such that all of these stable sets are disjoint.

We begin with $|A_1| + |B_1| - d(w) = |A_1| + |B_1| - \omega(G[A_1 \cup B_1])$ diads intersecting A_1 and B_1 . Since $G[A_1 \cup B_1]$ is cobipartite, these diads hit every vertex of $(A_1 \cup B_1) \setminus X_w$. Observe that $|A_1| + |B_1| - d(w) \geq t + d_{AB}$. So we want to extend some of the diads to triads. We can actually extend $|C_1|$ of them. To see this, note that there are at least three centres of degree $\geq |C_1|$ other than w , so every vertex in C_1 has at least $|C_1|$ non-neighbours in $(A_1 \cup B_1) \setminus X_w$. So we have $|A_1| + |B_1| - d(w) - |C_1|$ disjoint diads intersecting A and B and a further $|C_1|$ disjoint triads.

Thus it is clear that we can find the desired disjoint stable sets, beginning with the diads intersecting A and B . When picking our $d_{AC} + d_{BC}$ remaining diads we take a vertex in X_w not intersecting an AB diad whenever possible. Once we have found the necessary diads, we have enough AB diads remaining so that we can extend them to triads. These stable sets give us a $\chi(G')$ -colouring of G , contradicting the minimality of G .

Case 3: $d(w) < d(a)$.

As in the previous case, we remove edges from G_1 without introducing a claw or changing the chromatic number of G . There is at most one clique X in $G[B_1 \cup C_1]$ of size greater than $|B_1|$, by the structure of G_1 . If X exists, construct G' from G by removing all edges from G_1 except those within A_1 , B_1 , C_1 , and X . If such an X does not exist, set X as B_1 and construct G' from G by removing all edges from G_1 except those within A_1 , B_1 , and C_1 . It is easy to confirm that G' is claw-free and a proper subgraph of G . We will show that $\chi(G') = \chi(G)$, contradicting the minimality of G .

We claim that there is an $\omega(G_1)$ -colouring of G_1 using $|B_1| + |C_1| - |X|$ triads. To see this, we remove $|A_1| - |X|$ vertices from A_1 one at a time, always taking one from the largest clique X_s that still has a vertex in A_1 . If after removing k vertices we have disjoint X_s and $X_{s'}$ of size $|A_1| - k$, then we have $|A_1| + |B_1| + |C_1| \geq k + 2(|A_1| - k) + 2|C_1|$, contradicting the facts that there are at least four centres of degree $\geq |C_1|$ in H and that $|B_1| \leq |A_1| - k$. Thus we can see that we reach a perfect graph on $|X| + |B_1| + |C_1|$ vertices with clique number $|X|$. In an $|X|$ -colouring of this graph every colour class intersects both A_1 and X , thus the colouring uses exactly $|B_1| + |C_1| - |X|$ triads. The other colour classes are diads intersecting X and A_1 . Thus as in the previous case, we can rearrange the colour classes of a $\chi(G')$ -colouring \mathcal{C}' of G' to construct a $\chi(G')$ -colouring of G . \square

4.3.2 Completing the proof

We now combine our lemmas to prove Theorem 1.7.

Proof of Theorem 1.7. Let (G, A, B, C) be a minimum counterexample to the theorem. Then G is skeletal and is not an antiprismatic thickening. Theorem 4.1 tells us that (G, A, B, C) admits a hex-join into (G_1, A_1, B_1, C_1) and (G_2, A_2, B_2, C_2) such that (G_1, A_1, B_1, C_1) is in one of $\mathcal{TTC}_1, \mathcal{TTC}_2, \mathcal{TTC}_3, \mathcal{TTC}_5$ or \mathcal{TTC}_6 . Lemmas 4.6, 4.2, 4.3, 4.4, and 4.5 tell us that (G_1, A_1, B_1, C_1) cannot be in $\mathcal{TTC}_1, \mathcal{TTC}_2, \mathcal{TTC}_3, \mathcal{TTC}_5$ or \mathcal{TTC}_6 respectively. Thus G cannot exist, proving the theorem. \square

5 Compositions of strips and generalized 2-joins

In this section we describe how to colour claw-free *compositions of strips*, an important class of graphs built as a generalization of line graphs. For a discussion of this composition operation we refer the reader to [20] §5.2 or [4]. Rather than concerning ourselves with the global structure of these graphs, we instead

focus on decompositions that arise in these graphs, and how we might exploit these decompositions in order to extend partial colourings. These decompositions are *generalized 2-joins*:

Definition. Suppose vertex sets V_1 and V_2 partition $V(G)$ and there are cliques X_i and Y_i in V_i such that $X_1 \cup X_2$ and $Y_1 \cup Y_2$ are cliques, and there are no other edges between V_1 and V_2 . Then we say that $((X_1, Y_1), (X_2, Y_2))$ is a *generalized 2-join*.

Let G_1 and G_2 denote $G[V_1]$ and $G[V_2]$, respectively. In order to extend a $\gamma(G)$ -colouring of G_1 to a $\gamma(G)$ -colouring of G , merely having our generalized 2-join is not enough. Rather, we need to know the structure of G_2 and exploit properties of restricted colourings based on that structure. The structure of G_2 can be described in terms of *strips*.

Definition. A strip (G, A, B) is a claw-free graph G with two cliques A and B such that for any vertex $v \in A$ (resp. B), the neighbourhood of v outside A (resp. B) is a clique.

The strip (G, A, B) will actually be (G_2, X_2, Y_2) . We now examine five types of strips that will give us the five types of generalized 2-join that we need to deal with.

5.1 Five types of strips

The first strips we consider are linear interval strips, which are essential to the structure of quasi-line graphs. The other four types contain a W_5 , i.e. an induced C_5 with a universal vertex, and therefore can only appear in graphs that are not quasi-line.

5.1.1 Linear interval strips

Let G be a linear interval graph, and let cliques A and B be the $|A|$ leftmost and $|B|$ rightmost vertices of G in some linear interval representation of G . Then (G, A, B) is a *linear interval strip*.

5.1.2 Antihat strips

Let G be an antihat graph, and let G' be an antihat thickening, i.e. a thickening of $G \cup M$ under M as defined in Section 3.3. We specify two cliques of G' : $A' = I(A \setminus (X \cup \{a_0\}))$ and $B' = I(B \setminus (X \cup \{b_0\}))$. Then $(G' - I(a_0) - I(b_0), A', B')$ is a strip and if it contains a W_5 we say that it is an *antihat strip*. These antihat strips are a slight generalization of the antihat strips used in Chudnovsky and Seymour's survey [4].

5.1.3 Strange strips

Let H be a claw-free graph on cliques $A = \{a_1, a_2\}$, $B = \{b_1, b_2, b_3\}$, and $C = \{c_1, c_2\}$ with adjacency as follows: a_1, b_1 are adjacent; c_1 is adjacent to a_2 and b_2 and b_3 ; c_2 is adjacent to a_1 , a_2 , b_1 , and b_2 . All other pairs are nonadjacent. Let G be a thickening of H under $M = \{b_3c_1, b_2c_2\}$ (see Figure 3). Then $(G, I(A), I(B))$ is a strip and we say that it is a *strange strip*.

5.1.4 Pseudo-line strips

We will define a type of line graph and modify it slightly. Let J be a graph containing a path on vertices j_1, j_2, j_3 in order such that every edge of J is incident to at least one of j_1, j_2, j_3 . Let $H = L(J)$, and for $i \in \{1, 3\}$ let X_i be the set of vertices of H corresponding to edges incident to j_i in J . Both X_1 and X_3 are cliques. Let v_1 and v_2 be the vertices of H corresponding to the edges j_1j_2 and j_2j_3 respectively. Let G be a thickening of H under $M = \{v_1v_2\}$. Then (G, X_1, X_3) is a strip and if it contains a W_5 we say it is a *pseudo-line strip*.

These strips correspond to thickenings of the class \mathcal{Z}_3 defined in [7]. We call the vertices of J other than $\{j_i \mid 1 \leq i \leq 3\}$ *centres* of J .

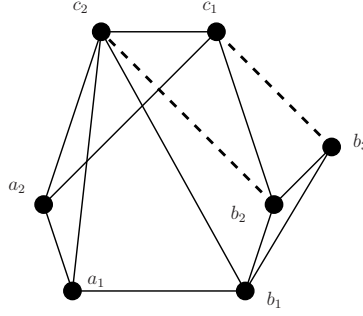


Figure 3: The graph underlying strange strips. Dashed lines represent edges in M .

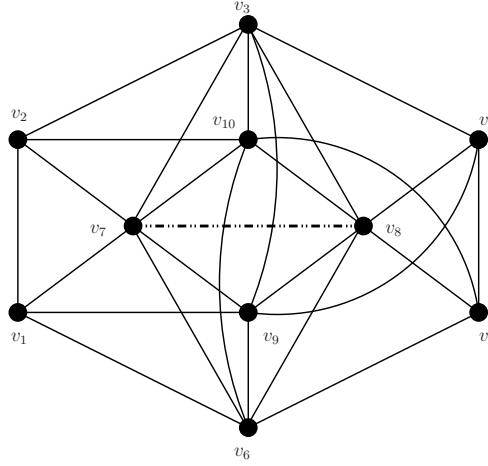


Figure 4: The graph underlying gear strips.

5.1.5 Gear strips

Take a graph H on vertices $\{v_1, \dots, v_{10}\}$ with adjacency as follows. The vertices v_1, \dots, v_6 are a 6-hole in order. Next, v_7 is adjacent to v_1, v_2, v_3, v_6 ; v_8 is adjacent to v_3, v_4, v_5, v_6, v_7 ; v_9 is adjacent to $v_3, v_4, v_6, v_1, v_7, v_8$; v_{10} is adjacent to $v_2, v_3, v_5, v_6, v_7, v_8$. There are no other edges in H . Let $X \subseteq \{v_9, v_{10}\}$. See Figure 4.

If G is a thickening of $H \setminus X$ under a matching $M \subseteq \{v_7v_8\}$, then $(G, I(v_1) \cup I(v_2), I(v_4) \cup I(v_5))$ is a strip, and we say that it is a *gear strip*, following the terminology of Galluccio, Gentile, and Ventura [18]. These correspond to thickenings of the class \mathcal{Z}_5 in [7] and are a slight generalization of thickenings of XX-strips as defined in [4].

5.2 Five types of generalized 2-joins

We now define generalized 2-joins corresponding to these five types of strips. Suppose in our claw-free graph G there is a generalized 2-join $((X_1, Y_1), (X_2, Y_2))$ separating G_1 and G_2 , such that X_1, X_2, Y_1 , and Y_2 are cliques and are pairwise disjoint except for possibly X_1 and Y_1 .

- **Canonical interval 2-joins.** If (G_2, X_2, Y_2) is a linear interval strip with X_2 and Y_2 disjoint such that G_2 is not a clique, then we say that $((X_1, Y_1), (X_2, Y_2))$ is a *canonical interval 2-join*.

- **Antihat 2-joins.** If (G_2, X_2, Y_2) is an antihat strip then we say that $((X_1, Y_1), (X_2, Y_2))$ is an *antihat 2-join*.
- **Strange 2-joins.** If (G_2, X_2, Y_2) is a strange strip then we say that $((X_1, Y_1), (X_2, Y_2))$ is a *strange 2-join*.
- **Pseudo-line 2-joins.** If (G_2, X_2, Y_2) is a pseudo-line strip then we say that $((X_1, Y_1), (X_2, Y_2))$ is a *pseudo-line 2-join*.
- **Gear 2-joins.** If (G_2, X_2, Y_2) is a gear strip then we say that $((X_1, Y_1), (X_2, Y_2))$ is a *gear 2-join*.

Aside from a limited set of exceptions, every claw-free graph that is not quasi-line or three-cliqued or an antiprismatic thickening admits one of these five types of generalized 2-join; we will state this more formally in the next section. We must now prove that no minimum counterexample to Theorem 1.6 admits any of these five types of generalized 2-join.

5.3 Dealing with the first four types

We begin by proving a lemma that implies that a minimum counterexample to Theorem 1.6 or Conjecture 1.2 cannot admit a canonical interval 2-join, an antihat 2-join, a strange 2-join, or a gear 2-join. Then we prove a lemma that implies that a that a minimum counterexample to Theorem 1.6 or Conjecture 1.1 cannot admit a pseudo-line 2-join. For each of these two tasks we need to define a special colouring invariant.

Given G admitting a generalized 2-join $((X_1, Y_1), (X_2, Y_2))$, let H_2 denote $G[V_2 \cup X_1 \cup Y_1]$ and denote $G_2 - X_2 - Y_2$ by Z_2 . For the first four types of generalized 2-join, we will use a local invariant $\gamma_\ell^j(H_2) \leq \gamma_\ell(G)$ which is easier to control when extending a partial colouring across a generalized 2-join. For pseudo-line 2-joins we will use an analogous global invariant $\gamma_g^j(H_2) \leq \gamma(G)$.

For a set of vertices S we define $\Delta_G(S)$ as $\max_{v \in S} d_G(v)$. Likewise we define $\omega(S)$ as $\max_{v \in S} \omega(v)$ and $\gamma_\ell(S)$ as $\max_{v \in S} \gamma_\ell(v)$. For $v \in H_2$ we define $\omega'(v)$ as the size of the largest clique in H_2 containing v and not intersecting both $X_1 \setminus Y_1$ and $Y_1 \setminus X_1$ (basically, $\omega'(v)$ is the largest clique containing v that we can easily manage). Let $\omega'(H_2)$ denote $\max_{v \in V(H_2)} \omega'(v)$. Now we define:

$$\gamma_\ell^j(H_2) = \max_{v \in V_2 \cup X_1 \cup Y_1} \left\lceil \frac{1}{2}(d_G(v) + 1 + \omega'(v)) \right\rceil.$$

$$\gamma_g^j(H_2) = \left\lceil \frac{1}{2}(\Delta_G(V(H_2)) + 1 + \omega'(H_2)) \right\rceil.$$

Observe that $\gamma_\ell^j(H_2) \leq \gamma_\ell(G)$ and $\gamma_g^j(H_2) \leq \gamma(G)$. Note that if $v \in X_1 \cup Y_1$, then $\omega'(v)$ is $|X_1| + |X_2|$, $|Y_1| + |Y_2|$, or $|X_1 \cap Y_1| + \omega(G[X_2 \cup Y_2])$. In [20] (and [2]) we proved:

Lemma 5.1. *Let G be a graph and suppose G admits a canonical interval 2-join $((X_1, Y_1), (X_2, Y_2))$. Then given a proper l -colouring of G_1 for any $l \geq \gamma_\ell^j(H_2)$, we can find a proper l -colouring of G in $O(nm)$ time.*

Since $\gamma_\ell^j(H_2) \leq \gamma_\ell(G) \leq \gamma(G)$ this lemma implies that no minimum counterexample to Theorem 1.6 or Conjecture 1.2 admits a canonical interval 2-join. We now prove a corresponding lemma for antihat, strange, and gear 2-joins in a skeletal claw-free graph.

Lemma 5.2. *Suppose a skeletal claw-free graph G admits a canonical interval 2-join or an antihat 2-join or a strange 2-join or a gear 2-join $((X_1, Y_1), (X_2, Y_2))$. Then given a proper l -colouring of G_1 for any $l \geq \gamma_\ell^j(H_2)$, we can find a proper l -colouring of G .*

As Lemma 5.1 deals with canonical interval 2-joins, we can split the remainder of the proof up into three lemmas corresponding to antihat 2-joins, strange 2-joins, and gear 2-joins. Our approach in each case is to set up the colouring of G_1 so that we can do one of two things. When possible, we colour G_2 directly by constructing an auxiliary graph from G_2 and appealing to perfection or Theorem 1.7. If that

is not possible then we remove stable sets, reducing $\gamma_\ell^j(H_2)$ each time, until G_2 becomes degenerate and we can appeal to a previous result.

Observe that it suffices to prove the case $l = \gamma_\ell^j(H_2)$. For if $l > \gamma_\ell^j(H_2)$, we can simply remove $l - \gamma_\ell^j(H_2)$ arbitrarily chosen colour classes and deal with what remains.

5.3.1 Antihat 2-joins

Lemma 5.3. *Suppose a skeletal claw-free graph G admits an antihat 2-join $((X_1, Y_1), (X_2, Y_2))$. Then given a proper l -colouring of G_1 for any $l \geq \gamma_\ell^j(H_2)$, we can find a proper l -colouring of G .*

Proof. Consider a minimum counterexample for some fixed l . If G_2 contains a skeletal homogeneous pair of cliques (A, B) then one of A and B is partially but not completely contained in one of X_2 or Y_2 .

Let k be the number of colours appearing in both X_1 and Y_1 . We begin by making k minimal, as we did in Case 6 of the proof of Lemma 5.1. To do this, we simply find a vertex v in $X_1 \cup Y_1$ with a colour appearing in both X_1 and Y_1 , such that some colour i does not appear in $X_1 \cup Y_1 \cup N(v)$, and recolour v with i . This minimality of k ensures a bound on l , as long as $k \geq 1$: Let vertices $u \in X_1$ and $v \in Y_1$ have the same colour. Then $d(u) + 1 \geq |X_2| + (l - |Y_1| + k)$, since minimality ensures that u has a neighbour in G_1 of every colour except possibly those in Y_1 not appearing in X_1 . Similarly, $d(v) + 1 \geq |Y_2| + (l - |X_1| + k)$. Therefore since $\omega'(u)$ and $\omega'(v)$ are at least $|X_1| + |X_2|$ and $|Y_1| + |Y_2|$ respectively, $l \geq |X_2| + \frac{1}{2}(l + k + |X_1| - |Y_1|)$ and $l \geq |Y_2| + \frac{1}{2}(l + k + |Y_1| - |X_1|)$. Consequently $l \geq |X_2| + |Y_2| + k$ if $k > 0$.

Suppose there is a colour class S in G_1 hitting X_1 but not Y_1 . Then add to this colour class a stable set S' of size two intersecting Y_2 and Z_2 . By the structure of antihat thickenings, we can assume that S' intersects $I(b_1)$ and $I(c_1)$ without loss of generality. If $\Omega(b_1 c_1)$ is nonempty, we insist that S' intersect it.

Note first that every vertex in $I(b_1) \cup I(c_1)$ is trumped or has a twin in S' or has two neighbours in S' . Every vertex in $Z_2 \setminus I(c_1)$ has two neighbours in S' , as does every vertex in $Y_2 \setminus I(b_1)$. Every vertex in X_2 has a neighbour in S and a neighbour in S' – this neighbour will be in Y_2 for vertices in $I(a_1)$, and in Z_2 for all other vertices in X_2 (recall from the definition of an antihat 2-join that since $I(b_1)$ and $I(c_1)$ are both nonempty, $I(a_1)$ is complete to $I(b_1)$ and anticomplete to $I(c_1)$). Thus $\gamma_\ell^j(v)$ drops for any $v \in G_2$. Since $S \cup S'$ intersects both $X_1 \cup X_2$ and $Y_1 \cup Y_2$, $\gamma_\ell^j(v)$ drops for any $v \in X_1 \cup Y_1$. Therefore we remove $S \cup S'$ and lower $\gamma_\ell^j(H_2)$.

We repeat this approach until either $Y_2 \cup Z_2$ is a clique, or all colours in X_1 appear in Y_1 . Suppose we remove t_1 stable sets in this way. We then take colour classes of G_1 hitting Y_1 but not X_1 , and remove them along with stable sets of size two in $X_2 \cup Z_2$, using the symmetric argument to show that $\gamma_\ell^j(H_2)$ drops each time. We do this until either all colours appearing in Y_1 are in X_1 , or until $X_2 \cup Z_2$ is a clique. Let t_2 be the number of stable sets we remove in this way, let S_1 be the set of all vertices we have removed from G , and let $t = t_1 + t_2$. Notice that $\gamma_\ell^j(H_2 - S_1) \leq \gamma_\ell^j(H_2) - t$.

Suppose $X_1 \setminus S_1$ is empty. Then we can colour $G_2 - S_1$ using $l - t$ colours by Theorem 1.7, since G_2 is three-cliqued. Since $Y_1 \setminus S_1$ is a clique cutset in $G - S_1$, this immediately gives us an $(l - t)$ -colouring of $G - S_1$ and therefore an l -colouring of G . So we can assume $X_1 \setminus S_1$ and symmetrically $Y_1 \setminus S_1$ are nonempty.

Now suppose every colour hitting $Y_1 \setminus S_1$ also hits $X_1 \setminus S_1$. Again we $(l - t)$ -colour $G_2 - S_1$, noting that at most $|X_2| + |Y_2| - t$ colours appear on $(X_2 \cup Y_2) \setminus S_2$ because $|(X_2 \cup Y_2) \setminus S_2| = |X_2| + |Y_2| - t$. We ensure that no colour hits both X_1 and X_2 , and that no colour hits both Y_1 and Y_2 . This is possible because $l - t > |(X_1 \cup X_2) \setminus S_1|$ and $l - t \geq |X_2| + |Y_2| + k - t$, as we proved above. This gives us a proper $(l - t)$ -colouring of $G - S_1$, and therefore an l -colouring of G .

By symmetry, this covers the case in which every colour hitting $X_1 \setminus S_1$ also hits $Y_1 \setminus S_1$. Thus there is a colour in X_1 but not Y_1 , and one in Y_1 but not X_1 . So our method stopped because both $(Y_2 \cup Z_2) \setminus S_1$ and $(X_2 \cup Z_2) \setminus S_1$ are cliques.

In this final case, we $(l - t)$ -colour $G_2 - S_1$ by applying Lemma 5.1 as follows. Notice that $(X_2 \setminus S_1, Y_2 \setminus S_1)$ is a homogeneous pair of cliques in $G - S_1$. We reduce it to a skeletal homogeneous pair

of cliques without changing the chromatic number using Theorem 2.1; the result is a graph G' in which $((X_1 \setminus S_1, Y_1 \setminus S_1), (X_2 \setminus S_1, Y_2 \setminus S_1))$ is a canonical interval 2-join. We can therefore apply Lemma 5.1 to find an $(l - t)$ -colouring of G' . Again using Lemma 2.3, we can construct an $(l - t)$ -colouring of $G - S_1$. This immediately gives us an l -colouring of G , proving the lemma. \square

5.3.2 Strange 2-joins

The next case is strange 2-joins; we use a similar approach.

Lemma 5.4. *Suppose a skeletal claw-free graph G admits a strange 2-join $((X_1, Y_1), (X_2, Y_2))$. Then given a proper l -colouring of G_1 for any $l \geq \gamma_\ell^j(H_2)$, we can find a proper l -colouring of G .*

Proof. As in the proof of the previous lemma, assume $\gamma_\ell^j(H_2) = l$ and let k denote the number of colours appearing in both X_1 and Y_1 . We begin by modifying the colouring of G_1 so that k is minimal, so again we can assume that either $k = 0$ or $l \geq |X_2| + |Y_2| + k$. Denote $G_2 - X_2 - Y_2$ by Z_2 .

Let $t = \min\{|I(a_1)|, |I(c_1) \cap \Omega(c_1, b_3)|, |Y_2| - k\}$. We remove t colours hitting Y_1 but not X_1 . With each colour class we remove a vertex of $I(a_1)$ and a vertex of $I(c_1) \cap \Omega(c_1, b_3)$. Together these vertices form t stable sets; call their union S_1 . As in the proof of the previous lemma, we now consider our situation depending on the value of t . Note that each time we remove a stable set, every vertex in G_2 is either trumped or loses two neighbours or loses a twin. It is therefore easy to see that $\gamma_\ell^j(H_2 - S_1) \leq \gamma_\ell^j(H_2) - t$.

Suppose $I(a_1)$ is empty. We apply Lemma 5.1 to $(l - t)$ -colour $G - S_1$ as follows. First observe that removing S_1 turns $((X_1 \setminus S_1, Y_1 \setminus S_1), (X_2 \setminus S_1, Y_2 \setminus S_1))$ into a *fuzzy linear interval 2-join*, meaning that we can turn it into a canonical interval 2-join by reducing nonskeletal homogeneous pairs of cliques: $(Z_2 \setminus S_1, Y_2 \setminus S_1)$ is a homogeneous pair of cliques in $G - S_1$, so we can reduce it to a skeletal homogeneous pair of cliques using Theorem 2.1, at which point $((X_1 \setminus S_1, Y_1 \setminus S_1), (X_2 \setminus S_1, Y_2 \setminus S_1))$ becomes a canonical linear interval 2-join in a proper claw-free subgraph G' of $G - S_1$. We can therefore apply Lemma 5.1 to G' , since we already have an $(l - t)$ -colouring of $G_1 - S_1$, to find an $(l - t)$ -colouring of G' . Theorem 2.1 tells us that we can use this colouring to construct an $(l - t)$ -colouring of $G - S_1$. Combining this with a t -colouring of $G[S_1]$ gives us an l -colouring of G .

Now suppose $I(c_1) \cap \Omega(c_1, b_3)$ is empty but $I(c_1)$ is not empty. To $(l - t)$ -colour $G - S_1$, we first remove the vertices of $I(b_3)$, which have become simplicial. Now observe that $((X_1 \setminus S_1, Y_1 \setminus S_1), (X_2 \setminus S_1, Y_2 \setminus (I(b_3) \cup S_1)))$ is an antihat 2-join. The remaining sets of G_2 are $I(a_1)$, $I(a_2)$, $I(b_1)$, $I(b_2)$, $I(c_1)$, and $I(c_2)$. To see the antihat 2-join, we relabel these sets as in the definition of an antihat thickening as follows: $(I(a_1), I(a_2)) \rightarrow (I(a_1), I(a_2))$, $(I(b_1), I(b_2)) \rightarrow (I(b_1), I(b_3))$, and $(I(c_1), I(c_2)) \rightarrow (I(c_3), I(c_1))$. We can therefore apply Lemma 5.3 to find an $(l - t)$ -colouring of $G - (S_1 \cup I(b_0))$, then replace and colour the simplicial vertices in $I(b_0)$ to get an $(l - t)$ -colouring of $G - S_1$. This gives us an l -colouring of G , completing the case of strange 2-joins. \square

5.3.3 Gear 2-joins

The final and most difficult case is that of gear 2-joins.

Lemma 5.5. *Suppose a skeletal claw-free graph G admits a gear 2-join $((X_1, Y_1), (X_2, Y_2))$. Then given a proper l -colouring of G_1 for any $l \geq \gamma_\ell^j(H_2)$, we can find a proper l -colouring of G .*

Proof. We proceed by induction on $|G|$, taking as our basis the trivial case in which $\min\{|X_1|, |Y_1|\} = 0$; in this case we have a 1-join and the result follows from Theorem 1.7 since gear strips are three-cliqued. So assume both X_1 and Y_1 are nonempty. Let Z_2 denote $G_2 \setminus (X_2 \cup Y_2)$. Again we can let G be a minimum counterexample and assume that $l = \gamma_\ell^j(H_2)$.

In this case we make k , the overlap between X_1 and Y_1 in the colouring of G_1 , maximal.

Case 1: $k > 0$.

If $k > 0$, we remove a colour class hitting both X_1 and Y_1 , along with one vertex each of $I(v_9)$ and $I(v_{10})$, if they are both nonempty. In this case every vertex of G_2 loses a twin or two neighbours. Since

we remove a vertex in both X_1 and Y_1 , it is easy to see that $\gamma_\ell^j(H_2)$ drops. Since removing vertices from $I(v_9)$ and $I(v_{10})$ will not change the fact that we have a gear 2-join, we can proceed by induction, having reduced both $\gamma_\ell^j(H_2)$ and l .

So assume that $I(v_9) \cup I(v_{10})$ is a clique, i.e. one of $I(v_9)$ and $I(v_{10})$ is empty. We do the same thing, but instead we remove a colour class hitting both X_1 and Y_1 , along with a vertex of $I(v_3)$ and a vertex of $I(v_6)$. Clearly $\gamma_\ell^j(H_2)$ drops as before and we can proceed by induction, since as long as neither $I(v_3)$ nor $I(v_6)$ becomes empty we will still have a gear 2-join.

Suppose $I(v_6)$ becomes empty, and one of $I(v_9)$ and $I(v_{10})$ is empty. By symmetry we can assume that $I(v_9)$ is empty. We are now left with a fuzzy linear interval 2-join: Reducing (if necessary) the possibly nonlinear homogeneous pairs of cliques $(I(v_7), I(v_8))$ and $(I(v_3) \cup I(v_{10}), I(v_4) \cup I(v_5))$ leaves us with a canonical interval 2-join. The vertices, in linear order, are $I(v_1)$, $I(v_2)$, $I(v_7)$, $I(v_3) \cup I(v_{10})$, $I(v_8)$, $I(v_4) \cup I(v_5)$. The reader can confirm this, along with symmetry between v_9 and v_{10} , by consulting Figure 4. So, as in the proof of the previous lemma, we can find our l -colouring of G by reducing on these two homogeneous pairs of cliques and invoking Lemma 5.1.

This completes the proof of the lemma when $k > 0$.

Case 2: $k = 0$; $l > |X_1| + |Y_1|$.

In this case we remove a colour class hitting neither X_1 nor Y_1 , along with a stable set of size three in G_2 . Call their union S . If $I(v_{10})$ is nonempty, we remove a vertex of $I(v_{10})$ along with one vertex each of $I(v_1)$ and $I(v_4)$. Every vertex in G_2 loses a twin or two neighbours, so it is easy to confirm that $\gamma_\ell^j(H_2)$ drops. Thus we can proceed by induction, provided that both $I(v_1)$ and $I(v_4)$ are still nonempty.

If $I(v_1)$ and $I(v_4)$ are both empty, then we extend the colouring of G_1 to an l -colouring of $G_1 \cup I(v_2) \cup I(v_5)$. We then note that $((I(v_2) \cup I(v_{10}) \setminus S, I(v_5) \cup I(v_{10}) \setminus S), (I(v_3) \cup I(v_7), I(v_6) \cup I(v_8)))$ is a fuzzy linear interval 2-join, in which $(I(v_3) \cup I(v_7), I(v_6) \cup I(v_8))$ is the only possible nonlinear homogeneous pair of cliques. So we can construct an $(l-1)$ -colouring of $G - S$ by Lemma 5.1 as in the previous two proofs. This gives us an l -colouring of S .

So assume $I(v_1)$ is now empty but $I(v_4)$ is not. Clearly we can extend the $(l-1)$ -colouring of $G_1 - S$ to a proper $(l-1)$ -colouring of $(G_1 - S) \cup I(v_2)$. We claim that we now have an antihat 2-join and we can find an $(l-1)$ -colouring of $G - S$ using Lemma 5.3.

The 2-join in $G - S$ is $((I(v_2), Y_1 \setminus S), ((I(v_3) \cup I(v_7)) \setminus S, Y_2 \setminus S))$. To see that $(G_2 - S) - (I(v_1) \cup I(v_2))$ is an antihat strip, we will relabel the vertices to conform with the definition of an antihat thickening. We relabel the sets $I(v_3)$, $I(v_{10})$, and $I(v_7)$ as $I(a_1)$, $I(a_2)$, and $I(a_3)$ respectively. We relabel $I(v_4)$ and $I(v_5)$ as $I(b_1)$ and $I(b_2)$ respectively. Finally, we relabel $I(v_6)$, $I(v_9)$, and $I(v_8)$ as $I(c_1)$, $I(c_2)$, and $I(c_3)$ (or $I(c_4)$ if $I(v_7) \cup I(v_8)$ is a clique) respectively. It is straightforward to confirm that this is an antihat strip. We therefore have an antihat 2-join in $G - S$, so by Lemma 5.3 we can find an $(l-1)$ -colouring of $G - S$ and an l -colouring of G .

If $I(v_{10})$ is empty, then instead of taking vertices from $I(v_{10})$, $I(v_1)$ and $I(v_4)$, we take vertices from $I(v_1)$, $I(v_3)$ and $I(v_5)$, and proceed symmetrically. This time, we may worry that $I(v_3)$ will become empty, but in this case, since $I(v_{10})$ is also empty, we get a fuzzy linear interval 2-join exactly as in Case 1.

Case 3: $k = 0$; $l = |X_1| + |Y_1|$.

In this final case, every colour appears in $X_1 \cup Y_1$, and no colour appears twice. Therefore X_2 and Y_2 must receive colours appearing in Y_1 and X_1 respectively. Since k is maximal, $l \geq |X_2| + |X_1| + \frac{1}{2}|Y_1|$ (from a vertex in X_1), and $l \geq |Y_2| + |Y_1| + \frac{1}{2}|X_1|$ (from a vertex in Y_1). It follows that $2l \geq \frac{3}{2}(|X_1| + |Y_1|) + |X_2| + |Y_2|$, so $|X_2| + |Y_2| \leq \frac{1}{2}l$.

Notice that Z_2 is cobipartite, and that the only non-edges in Z_2 are in $I(v_3) \cup I(v_6)$, $I(v_7) \cup I(v_8)$, and $I(v_9) \cup I(v_{10})$. We begin with an optimal colouring of Z_2 , removing the colour classes of size two. Let t_1 be the number of such colour classes in $I(v_3) \cup I(v_6)$, and let t be the total number of such colour classes. Denote these $2t$ vertices by S , noting that $Z_2 - S$ is a clique.

We construct an auxiliary graph G' from $G_2 - S$ by adding all possible edges between X_2 and Y_2 . Now G' is cobipartite and perfect, and since a proper colouring of G' will give vertices in X_2 and Y_2

distinct colours, it suffices to prove that $\omega(G') \leq l - t$. This gives us an l -colouring of G_2 in which no colour appears twice on $X_2 \cup Y_2$, so we can use it to extend the l -colouring of G_1 to an l -colouring of G .

Suppose there is a clique W of size greater than $l - t$ in G' . We will now prove that $l - |X_2| - |Y_2| \geq \frac{1}{2}|Z_2| \geq t$, which implies that W cannot be $X_2 \cup Y_2$. Consider vertices u, v, x, y in $I(v_1)$, $I(v_2)$, $I(v_4)$, and $I(v_5)$ respectively. Since every vertex in Z_2 has two neighbours in this set, the sum of the four degrees is at least $2(|X_1| + |X_2| + |Y_1| + |Y_2| + |Z_2|) - 4$. Therefore the sum $\gamma_\ell^j(u) + \gamma_\ell^j(v) + \gamma_\ell^j(x) + \gamma_\ell^j(y)$ is at least $4l \geq 2(|X_1| + |X_2| + |Y_1| + |Y_2|) + |Z_2|$. Thus $2l \geq |Z_2| + 2(|X_2| + |Y_2|)$, so $\frac{1}{2}|Z_2| + |X_2| + |Y_2| \leq l$.

A maximal clique W in G' intersecting both $I(v_1)$ and $I(v_2)$ as well as Z_2 must be $(I(v_1) \cup I(v_2) \cup I(v_7)) \setminus S$. But a vertex v in $(I(v_7) \cap \Omega(v_7, v_8)) \setminus S$ (this set is nonempty because $(I(v_7), I(v_8))$ is a skeletal homogeneous pair) has either two neighbours or a twin in each stable set of size two in S . This means that if $|W| > l - t$, then $\gamma_\ell^j(v) > l$, a contradiction. So W is not such a clique, and by symmetry W does not intersect all three of $I(v_4)$, $I(v_5)$, and $I(v_8)$. A similar argument implies that W cannot intersect only one of $I(v_1)$, $I(v_2)$, $I(v_4)$, and $I(v_5)$. Since $|X_2| + |Y_2| \leq l - t$ we can see that W cannot intersect three of these sets. Furthermore $|Z_2 - S| = \omega(Z_2) - t \leq l - t$, so W cannot be contained in $Z_2 - S$. Therefore W intersects all three of X_2 , Y_2 , and Z_2 , and we can assume by symmetry that W is $I(v_4) \setminus S$ and its neighbourhood in $X_2 \cup Y_2$, i.e. $(I(v_2) \cup I(v_3) \cup I(v_4)) \setminus S$.

Suppose that $|W| > l - t$. This inequality will provide us with new bounds on l , giving us a contradiction and completing the proof of the lemma. Let u and v be vertices in $I(v_2)$ and $I(v_4)$ respectively. Observe that $d(u) + 1 \geq |X_1| + |X_2| + |I(v_3) \setminus S| + t$, since u sees one vertex in every stable set in S . Thus $d(u) + 1 \geq |X_1| + |X_2| + |I(v_3)| + (t - t_1)$, and likewise $d(v) + 1 \geq |Y_1| + |Y_2| + |I(v_3)| + (t - t_1)$. Since $I(v_2) \cup I(v_3) \cup I(v_{10}) \cup I(v_7)$ is a clique, it follows that $\omega'(u) \geq |I(v_2)| + |I(v_3)| + (t - t_1)$, because every stable set of S hits $I(v_3) \cup I(v_7) \cup I(v_{10})$ exactly once. Likewise, $\omega'(v) \geq |I(v_4)| + |I(v_3)| + (t - t_1)$. The sum of these figures is at most $2\gamma_\ell^j(u) + 2\gamma_\ell^j(v)$, which is at most $4l$. This implies:

$$4l \geq (|X_1| + |Y_1|) + (|X_2| + |Y_2|) + 4(t - t_1) + 4|I(v_3)| + |I(v_2)| + |I(v_4)|.$$

We know that $|X_1| + |Y_1| = l$, $|X_2| + |Y_2| > |I(v_2)| + |I(v_4)|$, and by assumption, $|I(v_2)| + |I(v_4)| + |I(v_3)| - t_1 > l - t$. Therefore,

$$\begin{aligned} 3l &\geq 2(|I(v_2)| + |I(v_4)| + |I(v_3)|) + 2|I(v_3)| + 4(t - t_1) \\ &\geq 2l + 2|I(v_3)| + 2(t - t_1) \end{aligned}$$

Thus $|I(v_3)| - t_1 \leq \frac{l}{2} - t$. And since $|X_2| + |Y_2| \leq \frac{l}{2}$, we get $|W| = |I(v_2)| + |I(v_3)| + |I(v_4)| - t_1 \leq l - t$, contrary to our assumption.

It follows that $\omega(G') \leq l - t$, so we can indeed complete the l -colouring of G_2 that is compatible with the colouring of G_1 . This proves the lemma. \square

Lemmas 5.3, 5.4, and 5.5 together immediately imply Lemma 5.2.

5.4 Dealing with pseudo-line 2-joins

To deal with pseudo-line 2-joins we use $\gamma_g^j(H_2)$ rather than $\gamma_\ell^j(H_2)$.

Lemma 5.6. *Suppose a skeletal claw-free graph G admits a canonical interval 2-join or an antihat 2-join or a strange 2-join or a gear 2-join or a pseudo-line 2-join $((X_1, Y_1), (X_2, Y_2))$. Then given a proper l -colouring of G_1 for any $l \geq \gamma_g^j(H_2)$, we can find a proper l -colouring of G .*

Proof. We prove the lemma by induction on l . We let G be a minimum counterexample, noting that $l = \gamma_g^j(H_2)$. Assume that $|X_1| \geq |Y_1|$.

If $((X_1, Y_1), (X_2, Y_2))$ is a canonical interval 2-join or an antihat 2-join or a strange 2-join or a gear 2-join, then the lemma is immediately implied by Lemma 5.2 given the observation that $\gamma_\ell^j(H_2) \leq \gamma_g^j(H_2)$. So we can assume that we have a pseudo-line 2-join.

Recall that G_2 is based on the line graph of a graph J , and the vertices of J other than j_1 , j_2 , and j_3 are called *centres*. For a centre t in J , we call the corresponding clique C_t . That is, $C_t = \cup I(e)$ over all vertices e of H whose corresponding edge in J is incident to t . Let the edges j_1j_2 and j_2j_3 be e_1 and e_2 respectively. Note that Z_2 is a clique and so is $Z_2 \cup \Omega(e_1, e_2)$.

We begin by making the number k of colours in G_1 that hit both X_1 and Y_1 maximal. First suppose that there is no colour class appearing in neither X_1 nor Y_1 . As in the previous proofs, $l > |X_1|$. Since k is maximal, there is a vertex $v \in X_1$ with a colour not appearing in Y_1 , and it must have at least $l - 1$ neighbours in G_1 . This vertex is in $X_1 \cup X_2$, so $l = \gamma_g^j(H_2) \geq \frac{1}{2}l + \frac{1}{2}|X_1| + |X_2|$. Hence $l \geq |X_1| + 2|X_2|$. Since $l = |X_1| + |Y_1| - k$, we have $|X_2| \leq \frac{1}{2}|Y_1| - \frac{1}{2}k$. Now since $|X_2|$ is nonempty, $|Y_1| > k$ and there is a vertex in Y_1 with a colour not appearing in X_1 . We can therefore apply the symmetric argument to prove that $l \geq \frac{1}{2}l + \frac{1}{2}|Y_1| + |Y_2|$, and consequently $|Y_2| \leq \frac{1}{2}|X_1| - \frac{1}{2}k$.

Observe that if $|Z_2| \leq \frac{1}{2}(|X_1| + |Y_1|)$ we can easily finish the colouring by giving X_2 colours appearing in Y_1 but not X_1 , Y_2 colours appearing in X_1 but not Y_1 , and Z_2 colours appearing in both X_1 and Y_1 , and any leftover colours. In fact we can do this whenever $|Z_2| \leq l - |X_2| - |Y_2|$. So assume $|Z_2| > l - |X_2| - |Y_2|$. Let A be a maximum clique in $G[X_2 \cup Z_2]$. Since $G[X_2 \cup Z_2]$ is cobipartite, we can colour it with $|A|$ colours, $|X_2|$ of which intersect X_2 . Therefore if $|A| \leq l - |Y_2|$ we can colour Y_2 using colours that appear in X_1 but not in Y_1 , then colour X_2 and Z_2 using $|A|$ colours such that those colours appearing in X_2 do not appear in X_1 .

To see that $|A| \leq l - |Y_2|$, note that $\omega'(H_2) \geq |A|$ and since the degree of any vertex in $I(e_1)$ is at least $|X_1| + |X_2| + |Z_2| - 1$, $l = \gamma_g^j(H_2) \geq \frac{1}{2}(|A| + |Z_2| + |X_2| + |X_1|)$. Since $|Z_2| > l - |X_2| - |Y_2|$, this implies that $l > |A| + |X_1| - |Y_2| \geq |A| + \frac{1}{2}|Y_2|$. Therefore $|A| \leq l - |Y_2|$ and we can complete the $\gamma_g^j(G)$ -colouring of G .

We can now assume that there is a colour class S in G_1 that appears in neither X_1 nor Y_1 . We will find a stable set S_2 in G_2 such that removing $S \cup S_2$ lowers $\gamma_g^j(H_2)$; this will imply that $\chi(G) \leq l$ by induction.

First note that if there are at most two centres then we actually have an antihat 2-join – this is straightforward to confirm as there are only five vertices in J . So we can assume that there are at least three centres.

Suppose we set S_2 to be a diad (i.e. a stable set of size two) in $G[I(e_1) \cup I(e_2)]$ such that S_2 intersects $\Omega(e_1, e_2)$ if it is nonempty. S_2 exists because $G[I(e_1) \cup I(e_2)]$ is not a clique. If removing $S \cup S_2$ does not lower $\omega_j(G)$, then there must be a maximal clique in G_2 disjoint from S_2 . Such a clique must be C_t for some centre t that sees j_1 , j_2 , and j_3 in J .

The size of C_t must be at least $\max\{|X_1 \cup X_2|, |Y_1 \cup Y_2|, |Z_2|\} > \frac{1}{3}|V(G_2)|$, so by the number of vertices in G_2 there can be at most two such “centre cliques” of size $\omega'(H_2)$, since they must be disjoint – call the other one $C_{t'}$ if it exists. If we let S_2 be a stable set corresponding to a matching in J that hits three centres and in particular hits t and (if it exists) t' , we can see that removing $S \cup S_2$ lowers $\omega_j(G)$ so we are done. This S_2 must exist because C_t intersects all of X_2 , Y_2 , and Z_2 , so we can find S_2 unless every other centre has neighbourhood j_2 in J . If this is the case we can again easily confirm that we have an antihat 2-join, so we are done. \square

To prove Theorem 1.6, it only remains to deal with icosahedral thickenings.

6 Icosahedral thickenings

The icosahedron is the unique vertex-transitive graph on twelve vertices in which the neighbourhood of every vertex induces a C_5 . A result of Fouquet [16] tells us that a claw-free graph with $\alpha \geq 3$ is quasi-line precisely if no neighbourhood contains an induced C_5 , so the icosahedron is the epitome of a claw-free graph that is not quasi-line.

There are several graphs related to the icosahedron that we must treat as a structural exception, as they are not three-cliqued or antiprismatic, and they do not arise as a composition of strips, which we will define shortly. The first is the icosahedron itself, which we define explicitly. Let the graph G_0 have

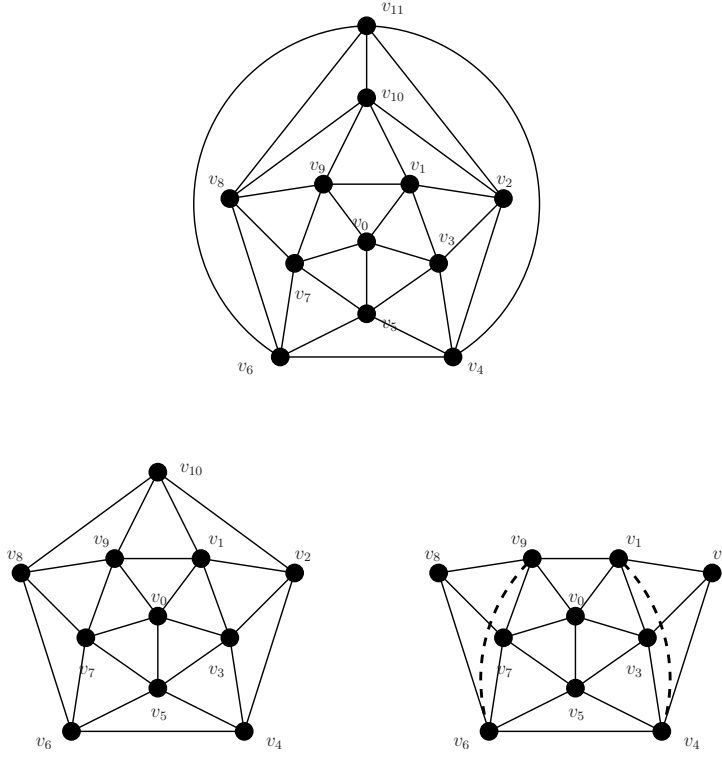


Figure 5: The icosahedron G_0 (top), with its derivative graphs G_1 (left) and $G_2 \cup M$ (right). In $G_2 \cup M$, each of $\{v_1, v_4\}$ and $\{v_6, v_9\}$ is a nonadjacent pair or is in M .

vertices v_0, v_1, \dots, v_{11} . For $i = 1, \dots, 10$, v_i is adjacent to v_{i+1} and v_{i+2} with indices modulo 10. The neighbourhood of v_0 is $\{v_i : 1 \leq i \leq 10, i \text{ is odd}\}$, and the neighbourhood of v_{11} is $\{v_i : 1 \leq i \leq 10, i \text{ is even}\}$. G_0 is the icosahedron (see Figure 5).

We obtain G_1 from G_0 by deleting v_{11} , and we obtain G_2 from G_1 by deleting v_{10} . Note that if the edge set M is a subset of $\{v_1v_4, v_6v_9\}$, then M is a claw-neutral matching in $G_2 \cup M$. We say that G' is an *icosahedral thickening* if it is a proper thickening of G_0 or G_1 , or is a thickening of $G_2 \cup M$ under some $M \subseteq \{v_1v_4, v_6v_9\}$. Any icosahedral thickening G' has $\alpha(G') = 3$ and $\chi(\overline{G'}) = 4$.

6.1 Colouring icosahedral thickenings

We now prove that any icosahedral thickening satisfies $\chi(G) \leq \gamma_\ell(G)$. To do so we remove triads from a supposed minimum counterexample, so first we need to consider induced subgraphs of icosahedral thickenings.

Lemma 6.1. *Let G be an icosahedral thickening. Then any skeletal induced subgraph G' of G is an icosahedral thickening or is three-cliqued or contains a clique cutset or admits a canonical linear interval 2-join.*

The proof of this lemma is straightforward but technical, and we leave it to the end of this section. This lemma allows us to prove the desired result:

Theorem 6.2. *Suppose G is an induced subgraph of an icosahedral thickening. Then $\chi(G) \leq \gamma_\ell(G)$.*

Proof. Let G be a minimum counterexample to the theorem. By Lemma 6.1 we know G is an icosahedral thickening or contains a clique cutset or is three-cliqued or admits a canonical interval 2-join. But G is

vertex-critical so it cannot contain a clique cutset. Lemma 5.1 (proved in [20] and [2]) and Theorem 1.7 tell us that G is in fact an icosahedral thickening.

First suppose that G is a proper thickening of G_0 , the icosahedron. We remark that the icosahedron is 4-colourable, so we remove four stable sets with union denoted by X containing exactly one vertex in $I(v_i)$ for every vertex v_i of G_0 . When X is removed, every remaining vertex v in G loses six neighbours (one of which is a twin), and since every maximal clique in G corresponds to a triangle in G_0 , $\omega(v)$ drops by three. Thus $d(v) + \omega(v)$ drops by nine and it follows that $\gamma_\ell(G)$ drops by at least four, contradicting the minimality of G .

Now suppose that G is a proper thickening of G_1 (see Figure 5). Again we remove one vertex from each $I(v_i)$, this time for $0 \leq i \leq 10$, again using four stable sets. When we remove the vertices, every remaining vertex loses at least five neighbours, one of which is a twin. And as with G_0 , every vertex v of G has $\omega(v)$ drop by three. Thus $\gamma_\ell(G)$ drops by at least four, contradicting the minimality of G .

Finally suppose that G is a thickening of $G_2 \cup M$ under a matching M ; we know that $M \subseteq \{v_1v_4, v_6v_9\}$. By minimality of G , $(I(v_1), I(v_4))$ and $(I(v_6), I(v_9))$ are skeletal homogeneous pairs of cliques. We remove two stable sets with union X : One intersects $I(v_1)$, $I(v_4)$, and $I(v_7)$ and intersects $\Omega(v_1v_4)$ if it is not empty. The other intersects $I(v_3)$, $I(v_6)$, and $I(v_9)$ and intersects $\Omega(v_6v_9)$ if it is not empty. These stable sets must exist because neither $I(v_1) \cup I(v_4)$ nor $I(v_6) \cup I(v_9)$ is a clique.

It is straightforward to confirm that X intersects every maximal clique in G , so $\omega(v)$ drops by at least one for every $v \in G - X$, thus $\gamma_\ell(v)$ drops by at least two for any vertex with three neighbours in X . Observe that any vertex in $G - X$ with only two neighbours in X must be in $(I(v_1) \cup I(v_4)) \setminus \Omega(v_1v_4)$ or $(I(v_6) \cup I(v_9)) \setminus \Omega(v_6v_9)$. Furthermore, every such vertex has a twin in X . Thus we can easily confirm that $\omega(v)$ drops by two for every such vertex. So for any v with only two neighbours in X , $\omega(v)$ drops by two. Therefore $\gamma_\ell(G - X) \leq \gamma_\ell(G) - 2$, contradicting the minimality of G . This completes the proof. \square

We now prove Lemma 6.1.

Proof of Lemma 6.1. Suppose first that G is a thickening of $G_2 \cup M$ under $M \subseteq \{v_1v_4, v_6v_9\}$ (see Figure 5). If G' has $I(v_i)$ nonempty for all $0 \leq i \leq 9$ then clearly G_2 is an icosahedral thickening unless $I(v_1) \cup I(v_4)$ or $I(v_6) \cup I(v_9)$ becomes a clique, in which case we have a clique cutset. If $I(v_i)$ is empty for some $i \in \{0, 2, 5, 8\}$ then it is not hard to check that G' is three-cliqued. If $I(v_i)$ is empty for some $i \in \{1, 4, 6, 9\}$ then G' contains a clique cutset. If none of these aforementioned sets $I(v_i)$ is empty but one of $I(v_3)$ and $I(v_7)$ is empty, then G' admits a canonical interval 2-join. For example, if G' is reached from G by deleting $I(v_3)$, then $((I(v_0) \cup I(v_9), I(v_5) \cup I(v_6)), (I(v_1), I(v_4)))$ is a canonical interval 2-join.

Now suppose that G is a thickening of G_1 . Obviously G' is an icosahedral thickening if $I(v_i)$ is nonempty for all $0 \leq i \leq 10$. If $I(v_i)$ is empty for any $i \in \{2, 4, 6, 8, 10\}$ then the desired result follows from the previous paragraph. If $I(v_0)$ is empty then G' is a circular interval graph. If $I(v_i)$ is empty for some $i \in \{1, 3, 5, 7, 9\}$ then it is easy to see from Figure 5 that G' admits a canonical interval 2-join or a clique cutset.

Finally, suppose that G is a thickening of G_0 . If G' has any $I(v_i)$ empty for $0 \leq i \leq 11$ then the desired result follows from the previous two paragraphs. Otherwise G' is clearly a thickening of G_0 . This completes the proof. \square

6.2 Proving the main result

6.2.1 A decomposition theorem

To prove Theorem 1.6 we use a decomposition theorem for claw-free graphs; it is a weakening of Theorem 7.2 in [7]:

Theorem 6.3. *Let G be a skeletal claw-free graph containing no clique cutset. Then one of the following is true:*

1. G is quasi-line

2. G is an antiprismatic thickening
3. G is three-cliqued
4. $\chi(\overline{G}) \geq 4$ and G admits a canonical interval 2-join, an antihat 2-join, a strange 2-join, a pseudo-line 2-join, or a gear 2-join
5. $\chi(\overline{G}) \geq 4$ and G is an icosahedral thickening.

Getting from Chudnovsky and Seymour’s structure theorem for claw-free trigraphs to Theorem 6.3 is complicated but not difficult. Still, we owe some explanation to the reader who is unfamiliar with trigraphs. First note that the structure of a graph is precisely the same as the structure of a trigraph in which no two (distinct) vertices are semiadjacent – only the terminology differs. The class of claw-free graphs is precisely the class of claw-free trigraphs in which no two vertices are semiadjacent. As a warm-up, one can easily check that if a claw-free graph G is a thickening of a trigraph G' , and G' is the union of three strong cliques, then G is a three-cliqued claw-free graph. Next, check that every graph which is a thickening of a member of \mathcal{S}_3 is quasi-line. Similarly, any graph which is a thickening of a member of \mathcal{S}_1 or \mathcal{S}_7 is an icosahedral thickening or an antiprismatic thickening, respectively.

This leaves *non-trivial strip structures*, discussed in Section 7 of [7]. Noting that in [7], $\mathcal{Z}_0 = \mathcal{Z}_1 \cup \dots \cup \mathcal{Z}_{15}$, observe that if a graph G is a thickening of a trigraph G' admitting a non-trivial strip structure involving a strip in $\mathcal{Z}_6 \cup \dots \cup \mathcal{Z}_{15}$, then G admits a clique cutset. Suppose now that G is a thickening of a trigraph G' admitting a non-trivial strip structure involving a strip in $\mathcal{Z}_2 \cup \dots \cup \mathcal{Z}_5$. Then G admits an antihat 2-join (arising from \mathcal{Z}_2), or a strange 2-join (arising from \mathcal{Z}_3), or a pseudo-line 2-join (arising from \mathcal{Z}_4), or a gear 2-join (arising from \mathcal{Z}_5). It now suffices to confirm that if G is a thickening of a trigraph G' admitting a non-trivial strip structure in which all strips are in \mathcal{Z}_2 or are trivial (i.e. (J, Z) where $|V(J)| = 3$ and $|Z| = 2$), then G is quasi-line.

6.2.2 Proof of Theorem 1.6

We can now combine our results to prove the second main result of the paper.

Proof of Theorem 1.6. Let G be a minimum counterexample to the theorem; clearly G cannot contain a clique cutset. Theorem 2.1 tells us that G is skeletal. Theorem 1.4 tells us that G is not quasi-line, Theorem 2.5 tells us that G is not an antiprismatic thickening, and Theorem 1.7 tells us that G is not three-cliqued. Lemma 5.6 tells us that G does not admit a canonical interval 2-join, an antihat 2-join, a strange 2-join, a gear 2-join, or a pseudo-line 2-join. Theorem 6.2 tells us that G is not an icosahedral thickening. Therefore G cannot exist. \square

7 Algorithmic considerations

We now show that our proofs of Theorems 1.6 and 1.7 yield polynomial time algorithms for $\gamma(G)$ - and $\gamma_\ell(G)$ -colouring G , respectively.

It is well known that we can restrict our attention to graphs containing no clique cutset – see e.g. [20] §3.4.3 for an explanation. By Theorem 2.1 we can restrict our attention to skeletal graphs. Furthermore we can identify maximal sets of twin vertices (i.e. equivalence classes of the “twin” equivalence relation) in G in polynomial time [10]. This immediately implies that we can recognize skeletal icosahedral thickenings in polynomial time. We can easily check whether or not a triad in a graph is good in polynomial time, so in polynomial time we can determine whether or not a graph contains a good triad by checking all triples of vertices.

If G is an icosahedral thickening, then observe that since G is skeletal there are at most 14 equivalence classes of twin vertices. Therefore there are at most 14^3 different types of stable sets. We can formulate the problem of colouring G as an integer program in which each variable represents the number of stable sets of a given type we use in the colouring. Each variable has size at most n , so we can exhaustively

solve the problem in $O(n^{14^3})$ time to find an optimal colouring of G (following the proof of Theorem 6.2 yields a much more efficient $\gamma_\ell(G)$ -colouring algorithm).

We now consider the problem of colouring three-cliqued claw-free graphs and antiprismatic thickenings.

7.1 Antiprismatic thickenings

We already know that any skeletal antiprismatic thickening contains a good triad, but we have not proven that reducing a nonskeletal homogeneous pair of cliques in an antiprismatic thickening leaves another antiprismatic thickening. It is enough to appeal to an easy result on *antiprismatic trigraphs*, which are defined in [7], Section 3. The proof is trivial but in the language of trigraphs.

Lemma 7.1. *If an antiprismatic trigraph G is a thickening of a trigraph H , then H is antiprismatic.*

Proof. Assume for a contradiction that either H contains a claw (in the trigraph sense) or that H contains four vertices among which at most one pair is strongly adjacent. In either case, the thickening from H to G provides us with a claw in G or a set of four vertices of G , among which at most one pair is strongly adjacent, a contradiction. \square

Corollary 7.2. *If (A, B) is a nonskeletal homogeneous pair of cliques in an antiprismatic graph G , and we obtain the graph G' by contracting A and B down to adjacent vertices a and b respectively, then G' is antiprismatic and G is a thickening of G' under a matching that contains ab .*

As a consequence of this corollary, reducing a nonskeletal homogeneous pair of cliques in an antiprismatic thickening will leave us with an antiprismatic thickening. We may therefore colour antiprismatic thickenings in the obvious way.

Theorem 7.3. *Given an antiprismatic thickening G , we can find a $\gamma_\ell(G)$ -colouring of G in polynomial time.*

Proof. Starting with G , we repeatedly apply Lemma 2.3, removing edges to reach a subgraph G' such that a k -colouring of G' gives us a k -colouring of G for any k . As we just showed, G' is an antiprismatic thickening, and therefore contains either no triad, in which case we can easily colour G' and therefore G in polynomial time, or contains a good triad T . In the latter case, we remove T and recursively $\gamma_\ell(G - T)$ -colour $G - T$, noting that $G - T$ is again an antiprismatic thickening.

Since we can perform the recursion steps in polynomial time and there are $O(m)$ possible steps, we can $\gamma_\ell(G)$ -colour G in polynomial time. \square

7.2 Three-cliqued graphs

Maffray and Preissmann proved that it is *NP*-complete to decide whether or not a triangle-free graph is three-colourable [23]. Consequently it is *NP*-complete to decide whether or not a claw-free graph is three-cliqued. This makes dealing with three-cliqued claw-free graphs a slightly delicate issue. However, consider a claw-free graph G . If $\alpha(G) \leq 2$ we know we can optimally colour it in polynomial time. We will show that if $\alpha(G) = 3$, then in polynomial time we can either $\gamma_\ell(G)$ -colour G , or determine that G is not three-cliqued.

Lemma 7.4. *Let G be a skeletal claw-free graph with $\alpha(G) = 3$, and suppose G contains no good triad. Then in polynomial time we can $\gamma_\ell(G)$ -colour G or determine that G is not three-cliqued.*

Proof. We define the *triad graph* $t(G)$ of G . We let $V(t(G)) = V(G)$, and two vertices are adjacent in $t(G)$ precisely if some triad in G contains both of them. We can easily find the components of $t(G)$ in polynomial time; there is at least one which is not a singleton.

Suppose first that G is three-cliqued. Then it admits a hex-join into terms (G_1, A_1, B_1, C_1) and (possibly empty) (G_2, A_2, B_2, C_2) such that G_1 is minimal and contains a triad. Since G contains no

good triad, it follows from the proofs of Lemmas 4.2, 4.3, 4.4, and 4.5 that (G_1, A_1, B_1, C_1) is in \mathcal{TTC}_1 . Furthermore the graph from which G_1 arises, i.e. H such that $G_1 = L(H)$, has more than three centres and hence more than six vertices, otherwise G would contain a good triad.

Suppose first that G is three-cliqued, and let X be a non-singleton component of $t(G)$. Then it admits a hex-join into terms (G_1, A_1, B_1, C_1) and (possibly empty) (G_2, A_2, B_2, C_2) such that G_1 is minimal and contains a triad. Since G contains no good triad, it follows from the proofs of Lemmas 4.2, 4.3, 4.4, and 4.5 that (G_1, A_1, B_1, C_1) is in \mathcal{TTC}_1 . Furthermore the graph from which G_1 arises, i.e. H such that $G_1 = L(H)$, has more than three centres and hence more than six vertices, otherwise G would contain a good triad.

We claim that there is a component X of $t(G)$ such that $X = V(G_1)$. First note that any component of $t(G)$ is either contained in $V(G_1)$ or disjoint from $V(G_1)$, since no triad can span both sides of a hex-join. Since every vertex of G_1 is in a triad, $V(G_1)$ is covered by non-singleton components of $t(G)$. In the case that G_1 contains a simplicial vertex v , it is easy to show that $V(G_1)$ is a component of $t(G)$: since every vertex is in a triad, every vertex not in $N(v)$ (in G_1) is in a vertex with v , and every vertex in $N(v)$ is in a triad, which is necessarily not contained in $N(v) \cup \{v\}$. So we may assume that G_1 contains no simplicial vertices.

Now it is sufficient to prove that every vertex in A_1 is in the same component of $t(G)$. Bearing in mind the structure of the bipartite multigraph H from which G_1 is constructed, the fact that G_1 has no simplicial vertex implies that the simple graph underlying H is a complete bipartite graph minus a matching. Therefore given two distinct edges of H incident to a , there must be two triads in G_1 containing their corresponding vertices, such that the triads intersect in two vertices. Therefore there is a component X of $t(G)$ such that $X = V(G_1)$.

For every component X of $t(G)$ we can test $G[X]$ for membership in \mathcal{TTC}_1 in polynomial time, because any graph in \mathcal{TTC}_1 is a proper thickening of a line graph of a specific bipartite graph H . In particular we can find (G_1, A_1, B_1, C_1) efficiently, because we can find H efficiently and the definition of \mathcal{TTC}_1 implies that the choice of vertices $\{a, b, c\}$ of H is unique. Thus since G_1 is a term in a hex-join, we can determine A_2 , B_2 , and C_2 by taking a vertex in G_2 and looking at its neighbourhood in G_1 , assuming that G is three-cliqued.

We now proceed as in the proof of Lemma 4.6. With our base graph H of (G_1, A_1, B_1, C_1) in hand, it is not hard to see that we can decide which action is necessary in polynomial time. In each case we find a triad whose removal is guaranteed to lower $\gamma_\ell(G)$ or we remove edges from G to reach a proper subgraph G' such that $\chi(G') = \chi(G)$. From the proof of Lemma 4.6 it is clear that we can find G' in polynomial time, and given a k -colouring of G' we can find a k -colouring of G in polynomial time. We can recursively $\gamma_\ell(G)$ -colour G' in polynomial time, possibly appealing to the fact that we can find good triads efficiently.

Now suppose G is not three-cliqued, which must be the case if no component of $t(G)$ induces a subgraph in \mathcal{TTC}_1 . If there is a component X of $t(G)$ such that $G[X]$ is in \mathcal{TTC}_1 , then again we have a unique choice of $\{a, b, c\}$ in H and a unique expression of $G[X]$ as (G_1, A_1, B_1, C_1) . Let A_2 be the set of vertices in $G - X$ which are complete to $A_1 \cup B_1$; we define B_2 and C_2 accordingly. Since G is not three-cliqued, either A_2 , B_2 , and C_2 do not partition the vertices of $G - X$, or they are not all cliques. Either way we can determine this in polynomial time. \square

Using these two lemmas we can prove the desired result:

Theorem 7.5. *Let G be a claw-free graph with $\alpha(G) \geq 3$. Then in polynomial time we can either $\gamma_\ell(G)$ -colour G or determine that $\chi(\overline{G}) \geq 4$.*

Proof. By Theorem 2.1 we can assume G is skeletal. If G contains a good triad T , we can find T in polynomial time and recursively $\gamma_\ell(G) - 1$ colour $G - T$, or determine that $\chi(\overline{G} - T) \geq 4$. If G does not contain a good triad, then the result follows immediately from Lemma 7.4. \square

7.3 Graphs that are not three-cliqued

By Theorem 6.3, if G is a skeletal claw-free graph that is not three-cliqued and does not contain a clique cutset, then one of the following applies:

1. G is an antiprismatic thickening
2. G is an icosahedral thickening
3. G is quasi-line
4. G admits a canonical interval 2-join or an antihat 2-join or a pseudo-line 2-join or a strange 2-join or a gear 2-join.

We already know how to deal with the first three cases efficiently, either by colouring in polynomial time or reducing to a smaller colouring problem. For each of the four latter types of generalized 2-join, of the form $((X_1, Y_1), (X_2, Y_2))$, there is a W_5 in G_2 whose neighbourhood contains G_2 . Given the correct choice of a W_5 in G , it is straightforward to find an appropriate generalized 2-join separating G_1 from G_2 in polynomial time (see [20] §8.2 for further details). There are $O(n^6)$ 5-wheels in G , so we can find such a generalized 2-join in polynomial time.

Since G is skeletal, we can easily check whether or not G_2 is a gear strip in polynomial time: a skeletal gear strip has at most twelve equivalence classes of twin vertices. So assume that we have an antihat 2-join or a pseudo-line 2-join or a strange 2-join. We can easily check for a strange 2-join similarly to checking for a gear 2-join. Checking if we have an antihat 2-join is straightforward once we determine the adjacency between X_2 and Y_2 . Otherwise we have a pseudo-line 2-join. In this case, $I(e_1)$ and $I(e_2)$ are precisely those vertices in X_2 and Y_2 respectively that are complete to $G_2 - X_2 - Y_2$. Furthermore, adding all edges between $I(e_1)$ and $I(e_2)$ leaves us with a line graph, the structure of which we can easily determine. Thus we can find these desired generalized 2-joins in polynomial time.

To reduce on these generalized 2-joins, we now consider the proof of Lemmas 5.3, 5.4, 5.5, and 5.6. We do one of two things: reduce the size of the graph and apply induction, or complete the l -colouring of G in one step. Just as with Lemma 5.1 in [21], showing that we can do this in polynomial time is straightforward given the proof of the lemma. Thus we get the desired algorithmic result:

Theorem 7.6. *For any claw-free graph G , we can $\gamma(G)$ -colour G in polynomial time.*

8 Proofs on homogeneous pairs of cliques

Finally, we give the postponed proofs of Lemmas 2.3 and 2.2.

8.1 Reducing on a nonskeletal homogeneous pair of cliques

We now prove Lemma 2.3, which is a straightforward generalization of Lemma 9 in [21]. This tells us exactly how we reduce on a nonskeletal homogeneous pair of cliques (A, B) and how we can manipulate colourings on (A, B) .

Proof of Lemma 2.3. Assume $|A| \geq |B|$. We can find a maximum clique X of $G[A \cup B]$ in $O(n^{5/2})$ time, choosing X to be A if A is a maximum clique. To construct G' from G , we remove precisely the edges between A and B that are not in X . Clearly $\omega(G'[A \cup B]) = \omega(G[A \cup B]) = |X|$, and (A, B) is a skeletal homogeneous pair of cliques in G' , so (1) holds. Since (A, B) is not skeletal, G' is a proper subgraph of G . We can find G' in $O(n^{5/2})$ time because we can find X in $O(n^{5/2})$ time [19].

We must prove that G' is claw-free. Suppose there is a vertex v seeing three mutually nonadjacent vertices a, b, c in G' . Then without loss of generality, $a \in A$, $b \in B$, and $c \notin A \cup B$ since G is claw-free. Since c sees neither a nor b in G' , c sees nothing in $A \cup B$ in G . It follows that $v \notin A \cup B$, so v sees all of $A \cup B$ in G . Therefore since A and B are not complete to each other in G , G contains a claw centred at v , a contradiction. So G' is claw-free.

Now suppose G is quasi-line; we must show that G' is quasi-line. Suppose a vertex v is not bisimplicial in G' and let (S, T) be a partitioning of $N_{G'}(v)$ into two cliques. If v has a neighbour $w \in S \setminus (A \cup B)$ that sees A but not B , then $B \subseteq T$ and thus $S \cup A$ and $T \setminus A$ are two cliques covering $N_{G'}(v)$ in G' . By symmetry we can assume that if no such w exists then all of $N_{G'}(v) \setminus (A \cup B)$ sees $A \cup B$, therefore $(S \cup A) \setminus B$ and $(T \cup B) \setminus A$ are two cliques covering $N_{G'}(v)$ in G' . Therefore G' is quasi-line if G is quasi-line. This proves (2).

Let $c_{G'}$ be a proper colouring of G' using $k \geq \chi(G')$ colours. Since (A, B) is a homogeneous pair, to construct a k -colouring of G it is enough to find a colouring of $G[A \cup B]$ that uses the same set of colours as $c_{G'}$ on A and on B . We can do this in $O(n^{5/2})$ time because the number of colours which appear on both A and B in the colouring of G' is at most the maximum size of a matching in $\overline{G'}$, which is the same as the size of a maximum matching in \overline{G} , i.e. $|(A \cup B) - X|$.

Since $G[A \cup B]$ is perfect, this extends to fractional colourings. Specifically, for any $l \geq \omega(G[A \cup B])$ there is a fractional l -colouring of $G[A \cup B]$. Suppose we have a fractional k -colouring of G' . This colouring uses weight $l \geq \omega(G[A \cup B])$ on $A \cup B$, so since (A, B) is a homogeneous pair of cliques we can combine the colouring of $G' - (A \cup B) = G - (A \cup B)$ with a fractional l -colouring of $G[A \cup B]$ to find a fractional k -colouring of G . This proves (3).

Suppose that G is three-cliqued. To prove (4), it suffices to prove that \overline{G} has a 3-colouring in which no colour appears in both A and B . If colour c_1 appears in both A and B then since $G[A \cup B]$ is not a clique, a second colour c_2 must appear in $A \cup B$; assume c_2 appears in A . In this case we can give all vertices of A colour c_2 and give all colours in B colour c_1 and since (A, B) is a homogeneous pair of cliques in G , the result is a valid 3-colouring of $\overline{G'}$. This proves (4). \square

8.2 Finding homogeneous pairs of cliques

Everett, Klein, and Reed gave a $O(mn^3)$ algorithm for finding homogeneous pairs [15], but did not consider the restricted case of homogeneous pairs of cliques.

In [21] we gave an $O(n^2m)$ -time algorithm for finding a nonlinear homogeneous pair of cliques; in [1] (Proposition 10) the same algorithm is shown to be implementable in $O(m^2)$ time, even in the setting of trigraphs.

Lemma 8.1. *For any graph G we can find a nonlinear homogeneous pair of cliques in G , or determine that none exists, in $O(m^2)$ time.*

Now we need to find linear nonskeletal homogeneous pairs of cliques. First we prove a structural result that renders the task almost trivial.

Lemma 8.2. *Suppose a graph G contains a nonskeletal linear homogeneous pair of cliques (A, B) . Then G contains three nonempty disjoint cliques A_1, A_2, B_1 such that*

- $|A_1| \geq |B_1|$.
- Each of A_1, A_2 , and B_1 is either a singleton or a homogeneous clique.
- $A_1 \cup A_2$ is a clique, $A_2 \cup B_1$ is a clique, and there are no edges between A_1 and B_1 .
- $(A_1 \cup A_2, B_1)$ is a nonskeletal linear homogeneous pair of cliques.

Proof. Suppose the vertices of $G[A \cup B]$ are $a_1, \dots, a_{|A|}, b_1, \dots, b_{|B|}$ in linear order.

By swapping the names of A and B , we can make an important assumption without loss of generality: Either A is a maximum clique in $G[A \cup B]$, or there is a maximum clique X of $G[A \cup B]$ and some vertex in B that sees some but not all of $X \setminus B$. If we cannot assume this, then $\omega(G[A \cup B]) > \max\{|A|, |B|\}$ and there is a unique maximum clique X in $(G[A \cup B])$. Furthermore since $(G[A \cup B])$ is a linear interval graph, no vertex in $A \setminus X$ (resp. $B \setminus X$) has a neighbour in B (resp. A), contradicting the assumption that (A, B) is nonskeletal.

To construct A_1, A_2 , and B_1 we first select two vertices a_p and a_q in A . Let p be minimum such that a_p is in a maximum clique X of $G[A \cup B]$; note that $p = 1$ if $\omega(G[A \cup B]) = |A|$. We claim that there is

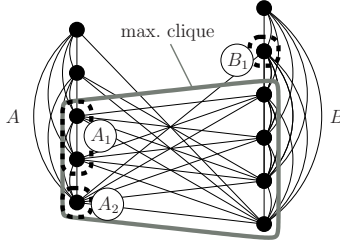


Figure 6: If a linear homogeneous pair of cliques is not skeletal, we can find within it a homogeneous pair of cliques with a very specific structure.

some minimum $q > p$ such that $\tilde{N}(a_p) \subset \tilde{N}(a_q)$, i.e. a_p and a_q are not twins. If q does not exist then by our above assumption either (i) $X = A$ and there are no edges between A and B , a contradiction since (A, B) is nonskeletal, or (ii) $|X| > |A|$ and no vertex in B sees some but not all of $X \setminus B$, a contradiction since in this case X must be the unique maximum clique of $G[A \cup B]$.

Let A_1 be a_p along with its twins, and let B_1 be the set of vertices that see that see a_p but not a_q . Clearly $B_1 \subseteq B$, and observe that $|A_1| \geq |B_1|$, otherwise a_p would not be in a maximum clique in $G[A \cup B]$, whereas a_q would. So let A_2 be q along with its twins. An example is shown in Figure 6.

To see that $(A_1 \cup A_2, B_1)$ is a homogeneous pair of cliques, it is enough to show that $(\{a_p, a_q\}, B_1)$ is a homogeneous pair of cliques. By the structure of linear interval graphs, every vertex in $A \setminus (A_1 \cup A_2)$ sees either all of B_1 or none of B_1 , so B_1 is a singleton or a homogeneous clique. Therefore $(\{a_p, a_q\}, B_1)$ is a homogeneous pair of cliques, following from the fact that (A, B) is a homogeneous pair of cliques. Furthermore since B_1 is complete to A_2 and anticomplete to A_1 , and $|A_1| \geq |B_1|$, it is easy to see that $(A_1 \cup A_2, B_1)$ is a nonskeletal linear homogeneous pair of cliques (in particular, $A_1 \cup A_2$ is a maximum clique in $G[A_1 \cup A_2 \cup B_1]$). \square

Thus when searching for a linear nonskeletal homogeneous pair of cliques, we can focus on this specific structure.

Lemma 8.3. *Let G be a graph containing no nonlinear homogeneous pair of cliques. Then in $O(nm)$ time we can find some nonskeletal linear homogeneous pair of cliques (A, B) in G , or determine that G is skeletal.*

Observe that Lemma 2.2 follows immediately from this lemma and Lemma 8.1.

Proof. We find a nonskeletal homogeneous pair of cliques (A, B) by finding the cliques A_1 , A_2 , and B_1 guaranteed by the previous lemma, as follows. First we partition the vertices of G into maximal homogeneous cliques in $O(m)$ time. After that we just need to find three vertices a_1 , a_2 , and b_1 inducing a path such that a_1 has at least as many twins as b_1 , no vertex sees a_1 but not a_2 , and b_1 and its twins are the only vertices that see a_2 but not a_1 . We can easily do this in $O(nm)$ time by first guessing b_1 , then deleting b_1 and checking for the appropriate resulting twins in $O(m)$ time. \square

Finally, we remark that we can find a skeletal homogeneous pair of cliques in $O(m)$ time. First we search for twins in time $O(m)$ – twins immediately lead to a homogeneous pair of cliques if the graph has at least four vertices. But the existence of a skeletal homogeneous pair (A, B) implies the existence of twins: Either $(A \cap \Omega(A, B), B \cap \Omega(A, B))$ is a homogeneous pair of cliques with all edges between them, or (A, B) is a homogeneous pair of cliques with no edges between them. Either case leads to twins. With the results of this section, this implies the following:

Theorem 8.4. *In $O(m^2)$ we can find a homogeneous pair of cliques in a graph or determine that none exists.*

9 Conclusion

The glaring open problem is Conjecture 1.2 for claw-free graphs. The only remaining case is that of compositions of pseudo-line strips, whose structure closely resembles that of line graphs. It is possible that a refinement of the approach taken in [2] would do the trick. For questions relating to more general local versions of the conjectures, we refer the reader to [13].

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References

- [1] M. Chudnovsky and A. D. King. Optimal antithickenings of claw-free trigraphs. Submitted. Arxiv preprint 1110.5111, 2011.
- [2] M. Chudnovsky, A. D. King, M. Plumettaz, and P. Seymour. A local strengthening of Reed’s ω , Δ , χ conjecture for quasi-line graphs. *SIAM J. Discrete Math.*, 2012. Accepted.
- [3] M. Chudnovsky and A. Ovetsky. Coloring quasi-line graphs. *J. Graph Theory*, 54:41–50, 2007.
- [4] M. Chudnovsky and P. Seymour. The structure of claw-free graphs. In B. S. Webb, editor, *Surveys in Combinatorics*, volume 327 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 2005.
- [5] M. Chudnovsky and P. Seymour. Claw-free graphs. I. Orientable prismatic graphs. *J. Comb. Theory Ser. B*, 97(6):867–903, 2007.
- [6] M. Chudnovsky and P. Seymour. Claw-free graphs. II. Non-orientable prismatic graphs. *J. Comb. Theory Ser. B*, 98(2):249–290, 2008.
- [7] M. Chudnovsky and P. Seymour. Claw-free graphs V. Global structure. *J. Comb. Theory Ser. B*, 98(6):1373–1410, 2008.
- [8] M. Chudnovsky and P. Seymour. Claw-free graphs VI. Colouring. *Journal of Combinatorial Theory, Series B*, 100(6):560 – 572, 2010.
- [9] V. Chvátal and N. Sbihi. Recognizing claw-free perfect graphs. *J. Comb. Theory Ser. B*, 44(2):154–176, 1988.
- [10] A. Cournier and M. Habib. A new linear algorithm for modular decomposition. In Sophie Tison, editor, *Trees in Algebra and Programming - CAAP’94, 19th International Colloquium, Edinburgh, U.K., April 11-13, 1994, Proceedings*, volume 787 of *Lecture Notes in Computer Science*, pages 68–84. Springer, 1994.
- [11] X. Deng, P. Hell, and J. Huang. Linear-time representation algorithms for proper circular-arc graphs and proper interval graphs. *SIAM Journal on Computing*, 25:390–403, 1996.
- [12] J. Edmonds. Paths, trees, and flowers. *Canadian J. Math.*, 17:449–467, 1965.
- [13] K. Edwards and A. D. King. A superlocal version of Reed’s conjecture. Submitted. Arxiv preprint 1208.5188, 2012.

- [14] F. Eisenbrand, G. Oriolo, G. Stauffer, and P. Ventura. Circular ones matrices and the stable set polytope of quasi-line graphs. In *IPCO XI, Lecture Notes in Computer Science 3509*, pages 291–305. Springer, 2005.
- [15] H. Everett, S. Klein, and B. A. Reed. An algorithm for finding homogeneous pairs. *Discrete Applied Mathematics*, 72(3):209–218, 1997.
- [16] J. L. Fouquet. A strengthening of Ben Rebea’s lemma. *J. Comb. Theory Ser. B*, 59:35–40, 1993.
- [17] T. Gallai. Über extreme Punkt-und Kantenmengen. *Ann. Univ. Sci. Budapest Eötvös Sect. Math.*, 2:133–138, 1959.
- [18] A. Galluccio, C. Gentile, and P. Ventura. Gear composition and the stable set polytope. *Oper. Res. Lett.*, 36(4):419–423, 2008.
- [19] J. E. Hopcroft and R. M. Karp. An $n^{5/2}$ algorithm for maximum matchings in bipartite graphs. *SIAM Journal on Computing*, 2:225–231, 1973.
- [20] A. D. King. *Claw-free graphs and two conjectures on ω , Δ , and χ* . PhD thesis, McGill University, October 2009.
- [21] A. D. King and B. A. Reed. Bounding χ in terms of ω and Δ for quasi-line graphs. *J. Graph Theory*, 59(3):215–228, 2008.
- [22] A. D. King, B. A. Reed, and A. Vetta. An upper bound for the chromatic number of line graphs. *Eur. J. Comb.*, 28(8):2182–2187, 2007.
- [23] F. Maffray and M. Preissmann. On the NP-completeness of the k -colorability problem for triangle-free graphs. *Discrete Math.*, 162:313–317, 1996.
- [24] F. Maffray and B. A. Reed. A description of claw-free perfect graphs. *J. Comb. Theory Ser. B*, 75:134–156, 1999.
- [25] M. Molloy and B. Reed. *Graph Colouring and the Probabilistic Method*. Springer, 2000.
- [26] T. Nielsen and J. Kind. The round-up property of the fractional chromatic number for proper circular arc graphs. *J. Graph Theory*, 33:256–267, 2000.
- [27] Gianpaolo Oriolo, Ugo Pietropaoli, and Gautier Stauffer. A new algorithm for the maximum weighted stable set problem in claw-free graphs. In Andrea Lodi, Alessandro Panconesi, and Giovanni Rinaldi, editors, *Integer Programming and Combinatorial Optimization*, volume 5035 of *Lecture Notes in Computer Science*, pages 77–96. Springer Berlin / Heidelberg, 2008.
- [28] K. R. Parthasarathy and G. Ravindra. The strong perfect-graph conjecture is true for $K_{1,3}$ -free graphs. *J. Comb. Theory Ser. B*, 21(3):212–223, 1976.
- [29] L. Rabern. A note on Reed’s Conjecture. *SIAM J. Discrete Math.*, 22(2):820–827, 2008.
- [30] B. A. Reed. ω , Δ , and χ . *J. Graph Theory*, 27:177–212, 1998.
- [31] W.-K. Shih and W.-L. Hsu. An $O(n^{3/2})$ algorithm to color proper circular arcs. *Discrete Applied Mathematics*, 25:321–323, 1989.